# On Compactness and Consistency in Finite Lattice-Valued Propositional Logic

**Xiaodong PAN and Yang XU**

*Southwest Jiaotong University, Chengdu, Sichuan, China*

**Luis MARTINEZ**

*Department of Computing, University of Jaén, E-23071 Jaén, Spain*

**Da RUAN**

*Belgian Nuclear Research Centre (SCK.CEN) and Ghent University, Belgium*

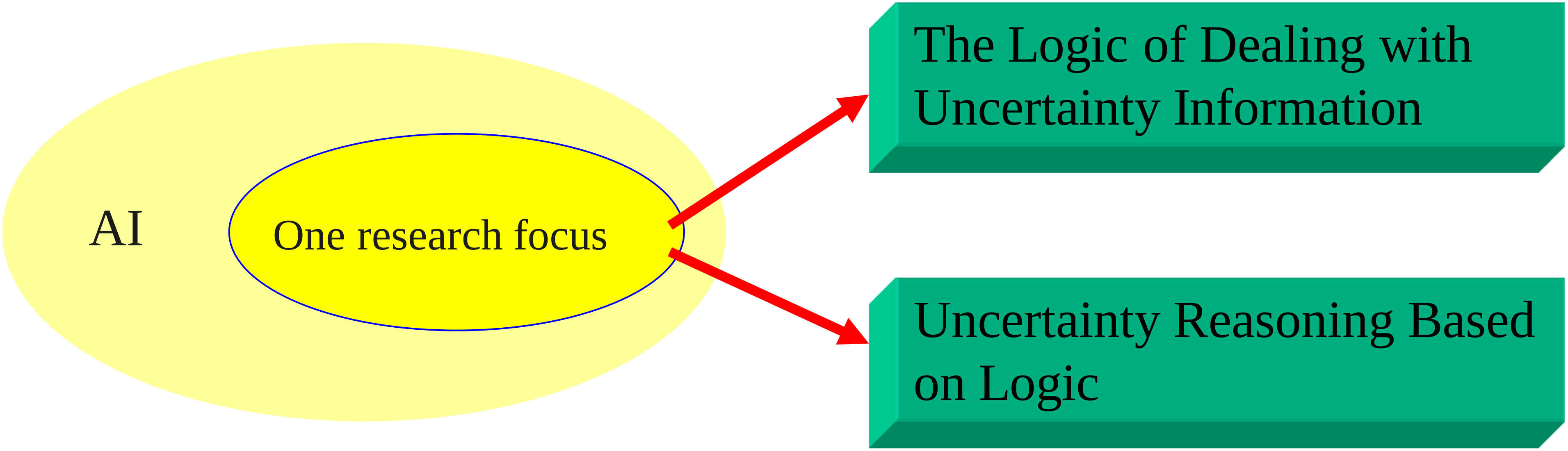
**Jun LIU**

*University of Ulster, Northern Ireland, UK*

# Outlines

- **Introduction**
- **Academic Background and Ideas**
- **Focused Technical Works**
- **Ongoing Research and Prospects**
- **Conclusion**

# Research View and Orientation




**Logic Based Intelligent Systems**

# Study of logic foundation for uncertainty reasoning: especially incomparability

- **Key ideas**

  Intelligent information processing → Uncertain Information → Uncertainty Reasoning → Need for establishing strict logic foundation → Non-Classical logic → Incomparable information → Lattice-valued logic system with truth-valued in a lattice

**Lattice + Logic**

- **Logical algebraic structure – lattice implication algebras (LIA)**

  Combining **lattice** and **implication** algebra, non-chain structure

- **Lattice-valued logic systems based on LIA**

  Incomparable information → Relation with fuzzy logic → Universal Algebra → Truth-valued attached → Syntax and semantics extension → Complete and Sound lattice-valued logic system

# Academic routine since 1993

- Lattice-valued logical algebra — **Lattice Implication Algebra** (LIA)
  - Y. Xu, Lattice implication algebra, *Journal of Southwest Jiaotong University* (in Chinese), 1993, 1, pp. 20-27.

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- **Structure and properties** of LIA
- **Lattice-valued algebraic logic** — lattice-valued logic based on LIA
- **Approximate reasoning** based on lattice-valued logic
- **Automated reasoning** based on lattice-valued logic

# A lattice-valued logical algebra -- lattice implication algebra (LIA)

**Definition** (LIA) Let ($L$, $\vee$, $\wedge$ , $'$) be a bounded lattice with an order-reversing involution " $'$ " and the universal bounds O, I, : $L\times L\to L$ be a mapping. ($L$, $\vee$, $\wedge$, $'$, $\to$) is called a **lattice implication algebra** (LIA) if the following conditions hold for all $x, y, z\in L$:

($I_1$) $x\to(y\to z)=y\to(x\to z)$ (exchange property)

($I_2$) $x\to x=\mathrm{I}$ (identity)

($I_3$) $x\to y=y'\to x'$ (contraposition or contrapositive symmetry)

($I_4$) $x\to y=y\to x=\mathrm{I}$ implies $x=y$ (equivalency)

($I_5$) $(x\to y)\to y=(y\to x)\to x$

($I_6$) $x\to(y\vee z)=(x\to y)\vee(x\to z)$ (implication $\vee$ -distributivity)

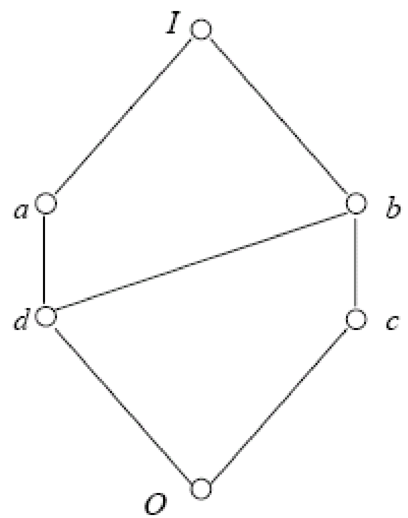
($I_7$) $x\to(y\wedge z)=(x\to y)\wedge(x\to z)$ (implication $\wedge$ -distributivity)

# Examples of LIA

Boolean algebra and Lukasiewicz algebra are all LIAs. A class of all LIAs form a proper class, which means many LIAs can be constructed and there are at least countable LIAs which can be constructed in [0, 1]

| $x$ | $x'$ |
|---|---|
| O | I |
| $a$ | $c$ |
| $b$ | $d$ |
| $c$ | $a$ |
| $d$ | $b$ |
| I | O |

| $\rightarrow$ | O | $a$ | $b$ | $c$ | $d$ | I |
|---|---|---|---|---|---|---|
| O | I | I | I | I | I | I |
| $a$ | $c$ | I | $b$ | $c$ | $b$ | I |
| $b$ | $d$ | $a$ | I | $b$ | $a$ | I |
| $c$ | $a$ | $a$ | I | I | $a$ | I |
| $d$ | $b$ | I | I | $b$ | I | I |
| I | O | $a$ | $b$ | $c$ | $d$ | I |




→ **Non-chain LIA**

# Book published (2003)

- Xu, Y., Ruan, D., Qin, K.Y., and Liu, J., *Lattice-Valued Logic – An Alternative Approach to Treat Fuzziness and Incomparability*, Springer-Verlag, Heidelberg, July, 2003, 390 pages.
- ISBN-3-540-40175-X

# The main focus of this paper: consistency and compactness in LP(X)

- Semantical theory of finite lattice-valued propositional logic LP(X) based on finite lattice implication algebras
- Semantical consequence operations in LP(X)
- Consistency and compactness in LP(X)

# LP(X), valuation and semantic consequence operation

By $LP$ we denote the lattice-valued propositional logic based on finite lattice implication algebras $L$. In $LP$, the formula set $\mathscr{F}$ is a $(\neg, \&, \Rightarrow)$-type free algebra generated by set $S \cup \bar{L}$, where $S$ is the set of propositional variables, $\bar{L} \triangleq \{\bar{a}|a \in L\}$, $\bar{a} \in \bar{L}$ is a nullary operation.

**Definition 2.** *The mapping* $v : \mathscr{F} \to L$ *is called a valuation if* $v(\neg A) = v(A)'$, $v(A\&B) = v(A) \otimes v(B) = (v(A) \to v(B)')'$, $v(A \Rightarrow B) = v(A) \to v(B)$, *and* $v(\bar{a}) = a$ *for any* $a \in L$.

The set $\mathscr{T}$ of all valuations is called the semantics of $LP$.

**Definition 3.** [14] *Let* $X \in L^{\mathscr{F}}, A \in \mathscr{F}, a \in L$. *The mapping* $\mathscr{C}_{\mathscr{T}} : L^{\mathscr{F}} \to L^{\mathscr{F}}, X \mapsto \bigwedge\{v \in \mathscr{T}|v \geqslant X\}$ *is called the L–semantic consequence operation on* $\mathscr{F}$.

*Remark 2.* Let $X \in L^{\mathscr{F}}$, $X$ is called an fuzzy theory on $\mathscr{F}$, for any $A \in \mathscr{F}$, $X(A) \in L$ denote the initial truth value of the formula $A$, it has been given in advance, so $X$ can also be viewed as an information qua premiss.

# True-valued level and satisfiability

In this section, we generalize the classical set of formulae to $L$-set in the level of $D$, discuss the properties of $L$-semantic consequence operation, and study the problem of the consistency. In what follows, we always admit that $\Gamma$ is a subset of $\mathscr{F}$ and $D$ be a subset of $L$ satisfying (i). $I \in D, O \notin D$; (ii). for any $x, y \in L$ such that $x \leqslant y$, $x \in D$ implies $y \in D$.

**Definition 4.** *Let $\Gamma \subset \mathscr{F}$, define $\Gamma_D \triangleq \{X \in L^{\mathscr{F}} | \forall A \in \mathscr{F}$, if $A \in \Gamma$ and $A \notin \bar{L}$, then $X(A) \in D$; if $A \in \Gamma$ and $A = \bar{\beta} \in \bar{L}$, then $X(A) = \beta$; otherwise, $X(A) = O\}$.*

*Remark 3.* In definition above, $\Gamma$ is a finite subset of $\mathscr{F}$ as usual. Let $X \in \Gamma_D$, for every $A \in \mathscr{F}$, by $X(A)$ we mean a truth value for $A$ in $X$ in the level of $D$, it have been given in advance. In fact, this can be viewed as a generalization of a premise formulas set in the classical semantic deduction.

**Definition 5.** *Let $\Gamma \subset \mathscr{F}$, $v \in \mathscr{T}$. $v$ is called as a model of $\Gamma$ in the level of $D$, or $v$ satisfies $\Gamma$ in the level of $D$ if $v(A) \in D$ for every $A \in \Gamma$ and $A \notin \bar{L}$. $\Gamma$ is called satisfiable in the level of $D$ if there exists a valuation $v$, which satisfies $\Gamma$ in the level of $D$.*

# Consistency and propositions

**Proposition 1.** *$\Gamma$ is satisfiable in the level of $D$ if and only if there exist $X \in \Gamma_D$ and $v \in \mathscr{T}$ such that $v \geqslant X$.*

**Definition 6.** *Let $X \in L^{\mathscr{F}}$, if $\{v \in \mathscr{T} | v \geqslant X\} = \emptyset$, then we assign $I_{\mathscr{F}}$ to $X$, i.e. $\mathscr{C}_{\mathscr{T}}X = I_{\mathscr{F}}$, where $I_{\mathscr{F}}$ is the greatest element of $L^{\mathscr{F}}$ which is the constant map equal to $I$ on the whole of $\mathscr{F}$. In this case, $X$ is said to be inconsistent with regard to $\mathscr{T}$; otherwise, $X$ is said to be consistent. If for all $X \in \Gamma_D$, $X$ is consistent, then $\Gamma$ is said to be consistent with regard to $\mathscr{T}$ in the level of $D$.*

**Proposition 2.** *$\Gamma$ is satisfiable in the level of $D$ if and only if there exists $X \in \Gamma_D$ such that $X$ is consistent with regard to $\mathscr{T}$.*

**Proposition 3.** *For any $X \in L^{\mathscr{F}}$, if there exist $A \in \mathscr{F}$ and $A \notin \bar{L}$ such that*

$$\left(\mathscr{C}_{\mathscr{T}}X\right)(A) \wedge \left(\mathscr{C}_{\mathscr{T}}X\right)(\neg A) \nleq \bigvee_{\alpha \in L} \left(\alpha \wedge \alpha'\right),$$

*then $X$ is inconsistent with regard to $\mathscr{T}$.*

# Additional properties

**Corollary 1.** *For any $X \in L^{\mathscr{F}}$, X is inconsistent with regard to $\mathscr{T}$ if and only if there exists $A \in \mathscr{F}$ such that*

$$\left(\mathscr{C}_{\mathscr{T}}X\right)(A) = \left(\mathscr{C}_{\mathscr{T}}X\right)(\neg A) = I.$$

In classical logic, by $M \vDash A$ we mean that $v(A) = 1$ for any $v \in \mathscr{T}$ satisfying $v(M) = \{1\}$. That is to say, $A \in \bigcap\{D_v | M \subset D_v, v \in \mathscr{T}\}$, where $D_v = \{p \in \mathscr{F} | v(p) = 1\}$. In the following, by the semantical consequence operation, we extend the above concept to lattice-valued logic, we can obtain the following conclusion:

**Theorem 1.** *Let $A \in \mathscr{F}$, if $\left(\mathscr{C}_{\mathscr{T}}X\right)(A) \in D$ for any $X \in \Gamma_D$, then $v(A) \in D$ for any $v \in \mathscr{T}$ satisfying $v(\Gamma) \subseteq D$.*

# Compactness of semantical consequence operation

Compactness is an important property of classical logic, which establishes a link between infinity and finity

A set of formula has a model if and only if every finite subset of it has a model

It provides a useful method for constructing models of any set of sentences that is finitely consistent

**Proposition 4.** *Let* $\mathscr{B}_{\mathscr{C}_{\mathscr{T}}} = \{Y \in L^{\mathscr{F}} \mid \mathscr{C}_{\mathscr{T}} Z \subseteq Y \text{ } for\ any\ Z \subseteq Y\}$, *then for any* $\mathscr{N} \subset \mathscr{B}_{\mathscr{C}_{\mathscr{T}}}$, $\bigwedge \mathscr{N} \in \mathscr{B}_{\mathscr{C}_{\mathscr{T}}}$; *that is to say,* $\mathscr{B}_{\mathscr{C}_{\mathscr{T}}}$ *is a closure system.*

**Corollary 2.** $\mathscr{B}_{\mathscr{C}_{\mathscr{T}}} = \{Y \in L^{\mathscr{F}} \mid \mathscr{C}_{\mathscr{T}} Y = Y\}$, *that is to say,* $\mathscr{B}_{\mathscr{C}_{\mathscr{T}}}$ *consists of all fixed points of* $\mathscr{C}_{\mathscr{T}}$.

**Corollary 3.** *For any* $Y \in L^{\mathscr{F}}$, $\mathscr{C}_{\mathscr{T}} Y = \bigwedge \{Z \mid Y \subseteq Z, Z \in \mathscr{B}_{\mathscr{C}_{\mathscr{T}}}\}$.

# Compactness relevant theorems

**Definition 7.** *The mapping $\mathscr{C}_{\mathscr{T}}$ is said to be compact if*

$$\mathscr{C}_{\mathscr{T}}X = \bigvee \left\{\mathscr{C}_{\mathscr{T}}Y \mid Y \in L^{\mathscr{F}}, Y \subseteq X \text{ and } Y \text{ is a finite fuzzy set}\right\}$$

*for any $X \in L^{\mathscr{F}}$.*

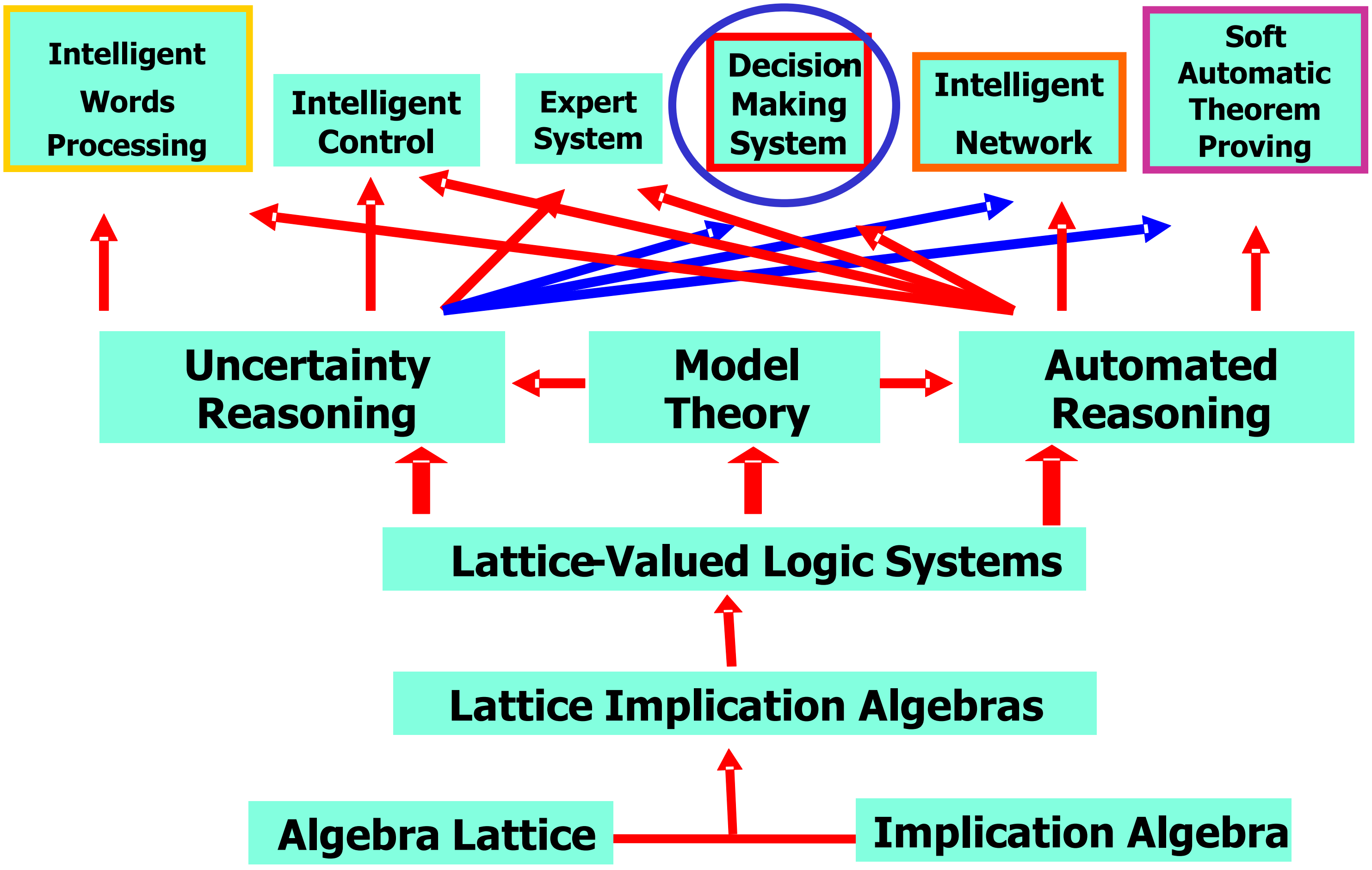
*$\mathscr{C}_{\mathscr{T}}$ is said to have the property of preserving directed joins if $\mathscr{C}_{\mathscr{T}}\left(\bigvee_{i\in I} Y_i\right) = \bigvee_{i\in I} \mathscr{C}_{\mathscr{T}}Y_i$ for any directed family $\mathscr{U} = \{Y_i \mid i \in I\}$ of subsets of $L^{\mathscr{F}}$, where $\mathscr{U}$ is said to be a directed family if for any $Y_i, Y_j \in \mathscr{U}$, there exists $Y_k \in \mathscr{U}$ such that $Y_i \subseteq Y_k$ and $Y_j \subseteq Y_k$.*

**Theorem 2.** *$\mathscr{C}_{\mathscr{T}}$ is compact if and only if it have the property of preserving directed joins.*

# Lattice-valued linguistic based automated reasoning and decision making

- **Representing linguistic terms**
  - Linguistic truth-value lattice-implication algebra
  - Linguistic atom term, logically composed terms, modified terms with a set of linguistic modifiers (hedges)
  - Their ordering relationship
  - Structure and characteristic
- **Lattice-valued linguistic resolution-based automated reasoning**
  - Structure and transformation, resolution principle, structure of resolution field, algorithm and programming
- **Application in decision making**

# A sketch map on research views, activities and directions



# Thank you !