The Lattice Computing (LC) information processing paradigm

(LC paradigm)

A Unifying Approach to Computational Intelligence

by

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INTRODUCTION

A Computational Intelligence (CI) technology is expected to be compatible with the principles of common sense. This book briefly presents a mathematical structure; the mathematical lattice for modeling logic. In particular, useful concepts and results from *lattice theory* (Birkhoff, 1967), which is alternatively known as *order theory*, are presented. Then, three different methodologies of extended CI from the literature, which are based on the theory of lattices, are summarized. Finally, the interest focuses on the grid of Interval Numbers (IN), which has been proposed by the authors of this book and here is constructively presented as a growing complexity hierarchy based on the set R of real numbers.

The set R is what is traditionally used to develop models (functions) of the form *f*: $R^N \rightarrow R^M$ based on measurements (Kaburlasos, 2006). For example, many laws of Physics, Economics, Engineering, etc. are successfully modeled in R^N. Note that a typical CI technology is used in practice as a mechanism for the implementation of a function of the form *f*: $R^N \rightarrow R^M$ (Kaburlasos & Kehagias, 2007). However, the transfer of modeling techniques from space R^N of numerical data to non-numerical data spaces that require new technologies, is not obvious, as described below

With the spread of PCs, it was created the need for large-scale processing of non-numeric data such as logical values, images, (fuzzy) sets, graphs, text, etc. with non-numeric representations and corresponding semantics. In an emerging new world, there is still the need for modeling, e.g. for recognition, prediction, etc. It is interesting that many of the (non) numerical data that appear in practice are partially ordered and are elements of a mathematical lattice. Note that recently the term **lattice computing** proposed as "a continuously evolving set of mathematical tools and mathematical modeling methodologies with the ability to handle partially-ordered data, as such, including logical values, numbers, sets, symbols, graphs etc.” (Esmi et al, in press · Graña & Chyzhyk, sub version · Kaburlasos & Papakostas, 2015 · Kaburlasos et al, 2013 · Sussner, sub version · Valle & Sussner, 2013). That is, *lattice computing* deals with partially-ordered data without transforming it into other kind of data, e.g. in real numbers.

Order theory is proposed here as a general modeling field, which, in addition to measurements, can also represent semantics. In particular, the innovative proposition here is the extension of the modeling field from the totally ordered set R to (partially ordered) lattices and ultimately, the calculation of a function of the form *f*: $L \rightarrow K$, where L and K are lattices. A comparative advantage of modeling in lattices is the representation of semantics with the partially ordered relation between the data and, ultimately, the calculation even with semantics instead of just the traditional calculation with numbers. In addition, note that since *information grains* are partially ordered (Liu et al., 2013; Sussner & Esmi, 2011), lattice theory could be used for analysis and design in *granular computing applications* (Pedrycz et al., 2008 · Zadeh, 1997).

The three chapters of Part-III present the following material. Chapter 7 presents basic concepts of lattice theory for use in the following chapters. Chapter 8 summarizes methodologies from the literature that include (1) Logic and Reasoning, (2) Formalistic

Concept Analysis and (3) Mathematical Morphology, where analysis and design are based on lattice theory. Finally, Chapter 9 focuses on the Study of Intervals' Numbers (IN), which describes how the partially ordered IN lattice can integrate popular semantic representations that include probability/feasibility distributions. Note that, from time to time, joint presentations of CI methodologies have been carried out in lattices (Kaburlasos, 2011; Kaburlasos & Ritter, 2007; Kaburlasos et al., 2008; Kaburlasos et al., 2014; Liu et al.)

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CHAPTER 7: LATTICE THEORY IN COMPUTATIONAL INTELLIGENCE

This chapter, in addition to the presentation of mathematical concepts and tools, proposes a unification in Computational Intelligence (CI) through the unification of dissimilar (partially ordered) data types that include reasonable values, numbers (and signals), (fuzzy) sets, graphs, strings, etc. within the framework of Lattice theory.

This chapter summarizes special mathematical tools of Lattice theory for CI applications. Infrastructure knowledge from Lattice theory is presented in the Appendix of this chapter to facilitate reading. Subsequently, the fuzzification of the binary relation “partial order” is modeled in a classic lattice.

7.1 Fuzzy Lattice

Consider a (classic) lattice $(\mathsf{L},\sqsubseteq)$, where for $x, y \in \mathsf{L}$ it is either $(x, y)\in\sqsubseteq$ or $(x, y)\notin\sqsubseteq$. In other words, the claim «$x \sqsubseteq y$» is either true or false, respectively. The calculation of a *degree of truth*, in some sense, of the «$x \sqsubseteq y$» assertion, is useful for making decisions in practical applications. For the aforementioned reason, the concept of *Fuzzy Lattice* is introduced, in order to fuzzify the relation “partial order” of a Lattice $(\mathsf{L},\sqsubseteq)$ so as to extend it to each pair $(x, y) \in \mathsf{L}\times\mathsf{L}$. In other words, the Fuzzy Lattice is a fuzzy set in the reference set $\mathsf{L}\times\mathsf{L}$.

A **Fuzzy Lattice** is defined as a triad $(\mathsf{L},\sqsubseteq,\mu)$, where $(\mathsf{L},\sqsubseteq)$ is a lattice and $(\mathsf{L}\times\mathsf{L},\mu)$ is a fuzzy set, such as $\mu(x, y)=1$ if and only if $x \sqsubseteq y$. It can be observed that the Fuzzy Lattice $(\mathsf{L},\sqsubseteq,\mu)$ fuzzifies the binary relation of this order to the lattice $(\mathsf{L},\sqsubseteq)$.

It should be noted that several authors have used the term "fuzzy lattice" in mathematics, with a different content (Ajmal & Thomas, 1994; Kehagias & Konstantinidou, 2003; Tepavčević & Trajkovski, 2001). However, the above definition is introduced in the context of mechanical-learning applications and classification in medical databases (Kaburlasos, 1992). Other authors have used the term fuzzy lattice to present the same concept in mathematics (Chakrabarty, 2001; Nanda, 1989), while in the Bayes probability theory the same concept was introduced with the name *zeta function* (Knuth, 2005).

As mentioned above, the motivation for the definition of a fuzzy lattice is the quantitative comparison of incomparable (e.g. parallel) lattice elements. In particular, to each pair $(x, y) \in \mathsf{L}\times\mathsf{L}$ is assigned a real number $\mu(x, y)\in[0,1]$ that indicates the degree to which *x* is less than or equal to *y*. When $x \sqsubseteq y$ then the fuzzy relation μ is valid between *x* and *y* to the maximum degree (i.e. 1), but the function μ may also be valid to a lower degree when $x \| y$ (i.e. when x and y are incomparable). Therefore, function μ can be interpreted as a **weak (fuzzy) partial order relation**. More specifically, function μ is characterized by a weak transient property so as if $\mu(x,y)=1$ and $\mu(y,z)=1$ are valid on the same time, then $\mu(x,z)=1$. However, if it is $\mu(x,y)\neq 1$, or $\mu(y,z)\neq 1$, then $\mu(x,z)$ can take any value in the interval [0,1].

It should be mentioned that the concept of Fuzzy Lattice is different from the *L-fuzzy set* concept (Goguen, 1967). The latter is a generalization of the concept of the *fuzzy set*, such as the membership function to depict the reference set in a complete lattice instead of depicting it exclusively in a conventional closed interval [0,1], as it does the classical theory of fuzzy sets.

Subsequently, a Fuzzy Lattice is calculated by introducing the function below (Kaburlasos & Papakostas, 2015):

Consider a lattice $(\mathsf{L},\sqsubseteq)$. **Fuzzy order** is defined as a function $\sigma:\mathsf{L}\times\mathsf{L}\rightarrow[0,1]$ that satisfies the two following properties:

C1. $u\sqsubseteq w \Leftrightarrow \sigma(u,w)=1$

C2. $u\sqsubseteq w \Rightarrow \sigma(x,u)\leq\sigma(x,w)$ (Consistency)

The fuzzy order function can be used to express the degree to which a lattice element is less than or equal to another element. Thus, the symbolism $\sigma(x\sqsubseteq y)$ can be used instead of $\sigma(x,y)$.

It is obvious that the property C1 requires the maximization of the fuzzy order $\sigma(u,w)=1$ when and only when the element u is less than or equal to element w. In other words, property C1 requires maximizing the fuzzy order exclusively for the elements that are arranged in the lattice $(\mathsf{L},\sqsubseteq)$. On the other hand, the property C2 requires some kind of *consistency* for a fuzzy order function in the following sense; if element u is contained in element w, then property C2 requests for any element x to be contained in u "no more" than it is contained in w.

Fuzzy order $\sigma(x\sqsubseteq y)$ is a generalization of alternative definitions proposed in the literature for the quantification of the degree of content of a (fuzzy) set in another (Fan et al., 1999; Sinha & Dougherty 1993; Sinha & Dougherty 1995; Young, 1996; Zhang & Zhang, 2009). However, the aforementioned alternative definitions refer only to overlapping pairs of (fuzzy) sets, otherwise the equivalent degree of content is equal to zero. Instead, the above definition of Fuzzy order is more general, because (a) it is applicable to each lattice and not only to the lattice of (fuzzy) sets, and (b) it concerns, without exception, all pairs of elements of any given lattice.

Every use of a fuzzy order function (σ) is called **Fuzzy Lattice Reasoning (FLR)** (Kaburlasos & Kehagias, 2014). In particular, a fuzzy order function (σ) supports two different types of reasoning, which is the *generalized reasoning* and *reasoning by analogy*. The *generalized reasoning* is a type of **deductive reasoning**, according to which, using the mathematical symbolism of lattice theory, given (a) a *logical rule* $a\rightarrow c$, and (b) a *reason* a_0 such as $a_0\sqsubseteq a$, follows a *result* c. On the other hand, *reasoning by analogy* is a type of

approximate reasoning, suitable for handling incomplete knowledge, as explained subsequently. Given (a) a set of rules $a_i \rightarrow c_i$, i∈{1,…,L}, and (b) a reason a_0 so as $a_0 \not\sqsubseteq a_i$, i∈{1,…,L}, the rule $a_J \rightarrow c_J$ that maximized the fuzzy order function $J \doteq arg_{i\in\{1,\ldots,L\}} \max \sigma(\alpha_0 \sqsubseteq \alpha_i)$ is selected. The result c_J, follows.

It has been proven that if function $v: L \rightarrow R_0^+$ is of positive valuation in lattice (L,⊑), then the two following functions (a) **Sigma-join**: $\sigma_\sqcup(x,u) = \frac{v(u)}{v(x \sqcup u)}$, και (β) **Sigma-meet**: $\sigma_\sqcap(x,u) = \frac{v(x \sqcap u)}{v(x)}$ are fuzzy order functions.

Assuming a lattice (L,⊑) and a fuzzy order $\sigma: L \times L \rightarrow [0,1]$. Then, the triad (L,⊑,σ) is fuzzy lattice. Therefore, the usability of having a fuzzy order (σ) on a lattice (L,⊑) is that it transforms the (L,⊑) into a fuzzy lattice and thus allows a quantified comparison of any two elements of (L,⊑), which can be either comparable or incomparable.

When a lattice (L,⊑) contains a minimum element *o*, a reasonable requirement is for the following equation to be valid: $\sigma_\sqcup(x,o) = 0 = \sigma_\sqcap(x,o)$, for every $x \sqsupset o$, i.e. the degree to which any (non-minimum) element is less than or equal to the minimum element *o* is zero. This requirement implies $v(o)= 0$.

7.2 Extensions in Lattice Hierarchies

This section deals with lattice hierarchies. Fuzzy order functions and metric functions are also introduced.

7.2.1 Cartesian Products of Lattices

A lattice (L,⊑) can be equal to the Cartesian product N – of possibly non-similar – **constituent** lattices (L_i,⊑), i=1,…,N. That is, it can be (L,⊑) = (L_1,⊑)×…×(L_N,⊑) where every basic lattice (L_i,⊑), i=1,…,N is characterized by its own order relation ⊑. For means of simplicity, the same symbols ⊑, ⊔, ⊓ are used in every lattice. For the same reason, the same symbols *o* και *i* are used for the least and greatest element of a complete lattice, respectively, unless it is differently indicated. The meet (⊓) and the join (⊔) operations in the lattice (L,⊑) = (L_1,⊑)×…×(L_N,⊑) are respectively calculated as follows:

$x \sqcap y = (x_1,\ldots,x_N) \sqcap (y_1,\ldots,y_N) = (x_1 \sqcap y_1,\ldots,x_N \sqcap y_N)$, και

$x \sqcup y = (x_1,\ldots,x_N) \sqcup (y_1,\ldots,y_N) = (x_1 \sqcup y_1,\ldots,x_N \sqcup y_N)$.

Moreover, it is true that $(x_1,\ldots,x_N) \sqsubseteq (y_1,\ldots,y_N) \Leftrightarrow x_1 \sqsubseteq y_1,\ldots,x_N \sqsubseteq y_N$.

If functions $v_1,\dots,v_N$ are valuations in the lattices $(L_1,\sqsubseteq),\dots,(L_N,\sqsubseteq)$, respectively, then function v: $L=L_1\times\dots\times L_N\to R$ given as $v= v_1+\dots+v_N$ is a valuation in lattice $(L=L_1\times\dots\times L_N,\sqsubseteq)$. If all valuations $v_1,\dots,v_N$ are *monotone function,* then valuation $v= v_1+\dots+v_N$ is also monotone. Additionally, if at least one of valuations $v_1,\dots,v_N$ is *positive* then valuation v is positive.

Metric functions and fuzzy order function are produced from positive valuation functions in (basic) lattices, as described below.

First, from a function v_i: $L_i\to R$ of positive valuation in the (basic) lattice $(L_i,\sqsubseteq)$, a metric function d_i: $L_i\to R_0^+$ is defined according to the equation $d_i(x_i,y_i)= v_i(x_1\sqcup y_1)-v_i(x_1\sqcap y_1)$, $i = 1,\dots,N$. Therefore, in the Cartesian product $(L,\sqsubseteq)= (L_1,\sqsubseteq)\times\dots\times(L_N,\sqsubseteq)$, a *metric Minkowski* function $d(;p): L\times L\to R_0^+$ is defined as follows:

$$d(x,y;p) = \left[\left(d_1(x_1,y_1)\right)^p+\dots+\left(d_N(x_N,y_N)\right)^p\right]^{1/p},\quad p\in R, x_i,y_i\in L_i \quad \text{with} \quad i = 1,\dots,N$$

where $x = (x_1,\dots,x_N)$, $y = (y_1,\dots,y_N)$.

Secondly, from positive valuation functions $v_1,\dots,v_N$ in (constituent) lattices $(L_1,\sqsubseteq),\dots,(L_N,\sqsubseteq)$, respectively, is defined a positive valuation function $v: L = L_1\times\dots\times L_N\to R_0^+$, defined as $v = v_1+\dots+v_N$. Thus, the following two fuzzy orders are resulted; $\sigma_\sqcup$: $L\times L\to[0,1]$ και $\sigma_\sqcap$: $L\times L\to[0,1]$, respectively.

Sigma-join: $\sigma_\sqcup(x,u) = \frac{v(u)}{v(x\sqcup u)} = \frac{v(u_1,\dots,u_N)}{v(x_1\sqcup u_1,\dots,x_N\sqcup u_N)} = \frac{\sum_{i=1}^N v_i(u_i)}{\sum_{i=1}^N v_i(x_i\sqcup u_i)}$, and

Sigma-meet: $\sigma_\sqcap(x,u) = \frac{v(x\sqcap u)}{v(x)} = \frac{v(x_1\sqcap u_1,\dots,x_N\sqcap u_N)}{v(x_1,\dots,x_N)} = \frac{\sum_{i=1}^N v_i(x_i\sqcap u_i)}{\sum_{i=1}^N v_i(x_i)}$

in the Cartesian product lattice $(L,\sqsubseteq)= (L_1\times\dots\times L_N,\sqsubseteq)$.

Alternatively, there is the following way to define a fuzzy order in the Cartesian product lattice $(L,\sqsubseteq)= (L_1\times\dots\times L_N,\sqsubseteq)$. In particular, given a positive valuation function $v_i: L_i\to R_0^+$ in the lattice $(L_i,\sqsubseteq)$, $i\in\{1,\dots,N\}$, according to all the above, two fuzzy order functions are resulted; (a) sigma-join $\sigma_\sqcup$: $L_i\times L_i\to[0,1]$, and (b) sigma-meet $\sigma_\sqcap$: $L_i\times L_i\to[0,1]$, respectively. Assuming that σ_i: $L_i\times L_i\to[0,1]$, $i\in\{1,\dots,N\}$ is a particular fuzzy order function of lattice $(L_i,\sqsubseteq)$, i.e. $\sigma_i(.,.)$ is either equal to $\sigma_\sqcup(.,.)$ or to $\sigma_\sqcap(.,.)$. Given non-negative real numbers $\lambda_1,\dots,\lambda_N$ such that $\lambda_1+\dots+\lambda_N = 1$, a fuzzy order σ_c: $L\times L\to[0,1]$ in the Cartesian product lattice $(L,\sqsubseteq)= (L_1\times\dots\times L_N,\sqsubseteq)$ is given by the **convex combination** $\sigma_c(x = (x_1,\dots,x_N), u = (u_1,\dots,u_N)) = \sum_{i=1}^N \lambda_i\,\sigma_i(x_i,u_i)$. Two other fuzzy orders are given form equations (a) $\sigma_\wedge(x,u)= \min_{i\in\{1,\dots,N\}}\sigma_i(x_i,u_i)$, and (b) $\sigma_\pi(x,u)= \prod_{i=1}^N \sigma_i(x_i,u_i)$, respectively.

7.2.2 Lattices of Intervals

Assuming ($\mathbb{I}$,$\subseteq$) is the partially ordered set (poset) of (ordinary) intervals of a lattice (L,$\sqsubseteq$). An interesting lattice is the ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$), where the meet operation ($\cap$) is defined as (1) $x\cap\emptyset=\emptyset$, for every $x\in(\mathbb{I}\cup\{\emptyset\})$, and (2) $[a,b]\cap[c,d] = [a\sqcup c,b\sqcap d]$ if $a\sqcup c\sqsubseteq b\sqcap d$, and $[a,b]\cap[c,d] = \emptyset$ if $a\sqcup c\not\sqsubseteq b\sqcap d$, for $[a,b],[c,d]\in\mathbb{I}$. While, the join operation ($\sqcup$) is defined as (1) $x\sqcup\emptyset=x$, for every $x\in(\mathbb{I}\cup\{\emptyset\})$, and (2) $[a,b]\sqcup[c,d] = [a\sqcap c,b\sqcup d]$, for $[a,b],[c,d]\in\mathbb{I}$. It should be noted that the lattice ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) is atomic since every interval $[a,b]\in\mathbb{I}$ is the join of two atoms; in particular, $[a,b] = [a,a]\sqcup[b,b]$. The atomic lattice ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) is a tool for analyzing complex data representations, as explained below

Of particular interest is the case where the lattice (L,$\sqsubseteq$) s complete with least and greatest elements o and i, respectively. In this case, the lattice (I_1=$\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) is also complete, with greatest element I=$[o,i]$. For the least element O which is the empty set ($\emptyset$), the representation O=$[i,o]$ is selected. An element of the set I_1= $\mathbb{I}\cup\{[i,o]\}$ is called **Type-1 (T1) interval**. Specific advantages of this representation O=$[i,o]$ are presented below.

An advantage of the representation O=$[i,o]$ is that the order relation $[a,b]\sqsubseteq[c,e]$, that is defined as $[a,b]\sqsubseteq[c,e] \Leftrightarrow$ «$c\sqsubseteq a$.KAI.$b\sqsubseteq e$», in lattice (I_1,$\sqsubseteq$) is compatible with the equivalence $[a,b]\subseteq[c,e] \Leftrightarrow c\sqsubseteq a\sqsubseteq b\sqsubseteq$e in the poset (I,$\subseteq$). An additional advantage of the representation O=$[i,o]$ is that the definitions of meet and join operations in lattice (I_1,$\sqsubseteq$) are consistent with the corresponding definitions in lattice ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) since (a) the meet ($\sqcap$) in lattice ($\mathsf{I}_1$,$\sqsubseteq$) is defined as $[a,b]\sqcap[c,e] = [a\sqcup c,b\sqcap e]$ if $a\sqcup c\sqsubseteq b\sqcap e$, and $[a,b]\sqcap[c,e] = [i,o]$ if $a\sqcup c\not\sqsubseteq b\sqcap e$, and (b) the join ($\sqcup$) in ($\mathsf{I}_1$,$\sqsubseteq$) is defined as $[a,b]\sqcup[c,e] = [a\sqcap c,b\sqcup e]$. Thus, in strict mathematical orology, we say that the lattices ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) and ($\mathsf{I}_1$,$\sqsubseteq$) are *isomorphic*, symbolized as ($\mathbb{I}\cup\{\emptyset\}$,$\subseteq$) $\approx$ (I_1, $\sqsubseteq$). Note that the lattice (I_1,$\sqsubseteq$) is also atomic.

Subsequently, metric and fuzzy order functions are investigated for lattice (I_1,$\sqsubseteq$). The aforementioned functions can be defined using a positive valuation function. However, a positive valuation function does not generally exist in lattice (I_1,$\sqsubseteq$), as shown in the anti-paradigm below.

Assume the intervals $[a, b]$, $[c, e]$, and $[c', e]$ in chain (R,$\leq$) of real numbers, where $a < b < c < c' < e$ as shown in Figure 7.1. It is obvious that $[c', e] < [c, e]$. Consider the following Initial Hypothesis: given a positive valuation function $v_1: (I \cup \{\emptyset\}) \times (I \cup \{\emptyset\}) \rightarrow R_0^+$. As a consequence of the aforementioned Initial Hypothesis, the following two equalities are valid:

1) $v_1([a, b]) + v_1([c, e]) = v_1([a, b] \sqcup [c, e]) + v_1([a, b] \sqcap [c, e]) = v_1([a, e]) + v_1(\emptyset)$, and

2) $v_1([a,b]) + v_1([c',e]) = v_1([a,b] \sqcup [c',e]) + v_1([a,b] \sqcap [c',e]) = v_1([a,e]) + v_1(\emptyset)$.

Thus, $v_1([c,e]) = v_1([c',e])$. However, this last equality appears to contradict the inequality $v_1([c,e]) < v_1([c',e]) \Leftarrow [c',e] < [c,e]$ that derives from the fact that function $v_1: (I \cup \{\emptyset\}) \times (I \cup \{\emptyset\}) \to R_0^+$ is of positive valuation. So, the Initial Hypothesis is not valid. In other words, a positive valuation function $v_1: (I \cup \{\emptyset\}) \times (I \cup \{\emptyset\}) \to R_0^+$ does not exist.

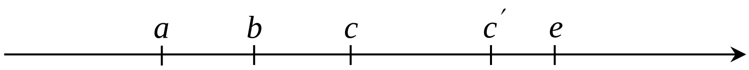

Fig. 7.1 *Five points $a < b < c < c' < e$ on the line of real numbers.*

According to the above, the investigation of metric and fuzzy order functions in the complete lattice $(\mathsf{I}_1,\sqsubseteq)$ of intervals T1, is performed in a different way, as described below.

Assume that the lattice $(\mathsf{I}_1,\sqsubseteq)$ can be embedded in a superlattice $(\mathsf{G},\sqsubseteq)$ in which a positive valuation function exist v_1: $\mathsf{G}\to R_0^+$ – The definitions of (order) embedded and superlattice are provided in the Appendix at the end of this Chapter. According to the above, the existence of function v_1: $\mathsf{G}\to R_0^+$ suggests the existence of a metric function (see Appendix) and of fuzzy order functions (see Section 7.1) in lattice $(\mathsf{G},\sqsubseteq)$. The above functions (i.e. metric and fuzzy order) are valid to the embedded lattice $(\mathsf{I}_1,\sqsubseteq)$.

Back to the complete lattice $(\mathsf{I}_1,\sqsubseteq)$ of intervals T1, which resulted from a complete lattice $(\mathsf{L},\sqsubseteq)$ with least and greatest elements *o* and *i*, respectively. Assume that in $(\mathsf{L},\sqsubseteq)$ there is a positive valuation function v: $\mathsf{L}\to R_0^+$ such that, as explained in Section 7.1 and in the Appendix of this Chapter, two **reasonable constraints** are valid; $v(o)=0$ και $v(i)<+\infty$. Due to the definition of order in a lattice $(\mathsf{I}_1,\sqsubseteq)$, the interest is focused on the Cartesian product of lattice $(\mathsf{L},\sqsubseteq^{\partial})\times(\mathsf{L},\sqsubseteq) = (\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)$ of the *generalized intervals* as a possible superlattice. A **generalized interval** is symbolized as $[a,b]$, $a,b\in\mathsf{L}$. The meet operation ($\sqcap$) in lattice $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)$ is calculated from $[a,b]\sqcap[c,d] = [a\sqcup c, b\sqcap d]$, while the join operation ($\sqcup$) in lattice $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)$ is calculated from $[a,b]\sqcup[c,d] = [a\sqcap c, b\sqcup d]$. The lattice$(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)$ is also complete with least and greatest elements $O=[i,o]$ and $I=[o,i]$, respectively.

In this Section it has been proven that the sum of positive valuation functions in each of the basic lattices $(\mathsf{L},\sqsupseteq)$ και $(\mathsf{L},\sqsubseteq)$ is a positive valuation function in Cartesian product $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)$. In the precious paragraph, it was assumed the existence of a positive valuation function v: $\mathsf{L}\to R_0^+$ in the complete lattice $(\mathsf{L},\sqsubseteq)$ in which two *reasonable constraints* exist; $v(o)=0$ and $v(i)<+\infty$. The goal now is to indicate a positive valuation function in $(\mathsf{L},\sqsupseteq)$. Note that the aforementioned positive valuation function v: $\mathsf{L}\rightarrow R_0^+$ in (L,⊑) is a valuation function in (L,⊒) where the operations ⊓ and ⊔ alternate, mutually. Thus, for $x,y\in(\mathsf{L},\sqsupseteq)$ it is valid that $v(x)+v(y) = v(x\sqcup y)+v(x\sqcap y)$. However, the valuation $v(.)$ is not positive in lattice (L,⊒) since $x\sqsubset y$ in (L,⊒) is equivalent to $x\sqsupset y$ in (L,⊑), and so $x\sqsubset y$ in (L,⊒) implies $v(x)>v(y)$.

The problem of the lack of a positive valuation function in lattice (L,⊒), given of a positive valuation function v: $\mathsf{L}\rightarrow R_0^+$ in lattice (L,⊑), can be resolved by assuming a (bijection) *dual isomorphism* function θ: $(\mathsf{L},\sqsubseteq)^{\partial}\rightarrow(\mathsf{L},\sqsubseteq)$, where $x\sqsubset y$ in (L,⊒) is equivalent to $\theta(x)\sqsubset\theta(y)$ in (L,⊑). Thus, the relation $x\sqsubset y$ in (L,⊒) is transformed in relation $\theta(x)\sqsubset\theta(y)$ in (L, ⊑) and, finally, it is implied that $v(\theta(x)) < v(\theta(y))$. In this way, the **composite function** $v\circ\theta(.)$ is a positive valuation function in (L,⊒) and, finally, function $v_1([x,y]) = v(\theta(x))+v(y)$ is a positive valuation function in ($\mathsf{L}\times\mathsf{L}$,⊒×⊑) of the generalized intervals. Additionally, since θ(.) is a bijection function, must apply $\theta(o)=i$ and $\theta(i)=o$. Consequently, the two *reasonable constraints* of a positive valuation function $v(.)$ are also valid for function $v_1(.)$. In particular, it is valid that, first, $v_1(\mathrm{O}=[i,o])= v(\theta(i))+v(o)= 2v(o)= 0$ and, secondly, $v_1(\mathrm{O}=[o,i])= v(\theta(o))+v(i)= 2v(i) < +\infty$.

Subsequently, the metric function d_1: $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\times(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\rightarrow R_0^+$, given from the type $d_1([a,b],[c,e])= v_1([a,b]\sqcup[c,e])- v_1([a,b]\sqcap[c,e])= v_1([a\sqcap c,b\sqcup e])- v_1([a\sqcup c,b\sqcap e])= v(\theta(a\sqcap c))+ v(b\sqcup e)- v(\theta(a\sqcup c))- v(b\sqcap e) = v(\theta(a)\sqcup\theta(c))-v(\theta(a)\sqcap\theta(c)) + v(b\sqcup e)-v(b\sqcap e)= d(\theta(a),\theta(c))+d(b,e)$, is also a metric function in lattice (I_1,⊑). Moreover, the fuzzy order function $\sigma_\sqcup$: $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\times(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\rightarrow[0,1]$, given from the type $\sigma_\sqcup([a,b],[c,e]) = \frac{v_1([c,e])}{v_1([a,b]\sqcup[c,e])}= \frac{v_1([c,e])}{v_1([a\sqcap c,b\sqcup e])}= \frac{v(\theta(c))+v(e)}{v(\theta(a\sqcap c))+v(b\sqcup e)}$ is also a fuzzy order function $\sigma_\sqcup$: $\mathsf{I}_1\times\mathsf{I}_1\rightarrow[0,1]$ in the embedder (sub)lattice (I_1, ⊑). Note that for $v_1([a,b]\sqcup[c,e]) = 0 \Leftrightarrow [a,b]\sqcup[c,e] = \varnothing \Leftrightarrow [a,b] = \varnothing = [c,e]$ it is considered by definition that $\sigma_\sqcup([a,b],[c,e]) = 1$. Finally, the fuzzy order function $\sigma_\sqcap$: $(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\times(\mathsf{L}\times\mathsf{L},\sqsupseteq\times\sqsubseteq)\rightarrow[0,1]$, given by the type $\sigma_\sqcap([a,b],[c,e]) = \frac{v_1([a,b]\sqcap[c,e])}{v_1([a,b])}= \frac{v_1([a\sqcup c,b\sqcap e])}{v_1([a,b])}= \frac{v(\theta(a\sqcup c))+v(b\sqcap e)}{v(\theta(a))+v(b)}$ is also a fuzzy order function $\sigma_\sqcap$: $\mathsf{I}_1\times\mathsf{I}_1\rightarrow[0,1]$ in the embedded (sub)lattice (I_1,⊑). Note that for $v_1([a,b]) = 0 \Leftrightarrow [a,b] = \varnothing$ it is considered by definition that $\sigma_\sqcap([a,b],[c,e]) = 1$.

Assuming a positive valuation function $v: L\rightarrow R$ in lattice (L,⊑), and (I,⊆) is the corresponding poset of the intervals. The non-negative function δ_1: $\mathsf{I}\rightarrow R_0^+$, calculated by the equation $\delta_1([a,b])= v(b)-v(a)$, is a *size* function of an interval since it satisfies the definition of size function. Note that $\delta_1([a,b]) = d(a,b) = \max_{x,y\in[a,b]} d(x,y)$, where $d: L\times L \to R_0^+$ is the metric $d(x,y)= v(x\sqcup y)-v(x\sqcap y)$ in lattice (L,⊑), i.e. the size of an interval $[a,b]$ is equal to the maximum distance of two elements x and y of the interval $[a,b]$. It can be observed that every **trivial interval** $[a,a]$ has zero *size*, i.e. $\delta([a,a])= 0$. Moreover, every trivial interval $[a,a]$ is *atom* in the complete lattice $(I_1,\sqsubseteq)$ of intervals T1 since it covers the least element o= ∅ in $(I_1,\sqsubseteq)$.

7.2.3 Sets of Lattice Elements

Suppose the powerset 2^L of a lattice (L,⊑). According to the previous sections, note that L may be the Cartesian product of N basic lattices, including interval lattices.

The binary relation ⊑ is defined in the Cartesian product $2^L \times 2^L$ so as for $U, W \in 2^L$ to be $U \sqsubseteq W$ if and only if $\forall u \in U, \exists w \in W: u \sqsubseteq w$. Thus, it results the lattice $(2^L, \sqsubseteq)$ with $U \sqcap W = \bigcup_{u\in U, w\in W}\{u \sqcap w\}$ and $\sqcup W = \bigcup_{u\in U, w\in W}\{u \sqcup w\}$.

A subset S of L is called **simplified** or, alternatively, **quotient**, if S contains only incomparable data of L. A non-simplified set S is called *simplifiable* and can be simplified if every subset of comparable elements $X \subseteq S$ is replaced with the least upper bound $\vee X$. Assume that $\pi(2^L) \subseteq 2^L$ is a subset of 2^L that contains only all the simplified subsets L. The interest is focused in elements of the lattice $(\pi(2^L), \sqsubseteq)$ since every element of $\pi(2^L)$ maximizes any fuzzy order function σ: $2^L\times 2^L \to [0,1]$ due to the property C2 (Consistency). Moreover, the fuzzy order function $\sigma_c: \pi(2^L)\times\pi(2^L) \to [0,1]$ in the lattice $(\pi(2^L), \sqsubseteq)$ can be defined as follows:

Suppose a fuzzy order function $\sigma: L\times L \to [0,1]$ in lattice (L,⊑). Then, function $\sigma_c: \pi(2^L)\times\pi(2^L) \to [0,1]$ which is given from the convex combination $\sigma_c(U\leq W)= \sum_{i=1}^{N} \lambda_i \max_{j\in\{1,\dots,N\}} \sigma(u_i \sqsubseteq w_j)$ is of fuzzy order, where $U=\{u_1,\dots,u_M\}$, $W=\{w_1,\dots,w_N\}\in\pi(2^L)$.

7.3 Unification of Dissimilar Data Types

This section presents specific examples of lattices within the previous sections. Each lattice concerns a different type of *data* that is ordered. In the above-mentioned way, the lattice theory unifies dissimilar types of data. Here it is considered that fuzzy order relation represents the semantics of the data. In this sense, lattice computing is a calculation with semantics.

7.3.1 Real Numbers

The totally ordered lattice (R,≤) of real numbers, the most popular of all lattices, along with its extensions, is analyzed extensively in Chapter 9. At this point it is only noted that a **strictly increasing function** v: R→R is a positive valuation function in lattice (R,≤). Moreover, a **strictly decreasing function** θ: R→R is a function of dual isomorphism in lattice (R,≤). Extension of the lattice (R,≤) in Cartesian product, partially ordered lattice $(R^N, \sqsubseteq) = (R,\leq)^N$, will be mentioned in this section.

By choosing the same positive valuation function $v(x)= x$ in every dimension of the lattice (R^N,⊑), results the metric $d(\boldsymbol{x},\boldsymbol{y};p)= [(d(x_1,y_1))^p+\ldots+(d(x_N,y_N))^p]^{1/p}= [|x_1-y_1|^p+\ldots+|x_N-y_N|^p]^{1/p}$, where $\boldsymbol{x}=(x_1,\ldots,x_N)$ και $\boldsymbol{y}=(y_1,\ldots,y_N)$, known in the bibliography as $\boldsymbol{L_p}$ **metric**. In particular, L_1 ιis equal to $d(x,y;1)=|x_1-y_1|+\ldots+|x_N-y_N|$ and it is known as **Hamming distance (**or **city-block distance)**, L_2 is the **Euclidean distance** $(x,y;2)=\sqrt{(x_1-y_1)^2+\ldots+(x_N-y_N)^2}$, while L_∞ is equal to $d(x,y;\infty)=max\{|x_1-y_1|,\ldots,|x_N-y_N|\}$.

An additional extension results if it is considered the Cartesian product R×…×R which corresponds to a non-countable set of basic lattices σ (R,≤). In this case, it results the partially ordered lattice (F,⊑) of all real functions defined on set R of real numbers. More specifically, given $f,g\in$F, the relation $f \sqsubseteq g$ is interpreted as $f(x) \leq g(x)$, for every $x \in R$, where «≤» is the order relation of (real) numbers. The meet operation (⊓) of two elements (of functions) f and g in lattice (F,⊑) is defined as $f \sqcap g = f(x) \sqcap g(x) := \min_{\forall x\in R}\{f(x),g(x)\}$, while the join operation (⊔) in lattice (F,⊑) is defined as $f \sqcup g = f(x) \sqcup g(x) := \max_{\forall x\in R}\{f(x),g(x)\}$.

7.3.2 Measure Spaces

As **measure space** is defined a triad $(\Omega,\Sigma_\Omega,m_{\Sigma_\Omega})$, where Ω is a set, Σ_Ω is a σ-algebra of set Ω, and m_{Σ_Ω} is a measure on Σ_Ω. Definitions of σ-algebra and measure are provided below.

As **σ-άλγεβρα** Σ_Ω of a set Ω is defined a collection of subsets of Ω such that the following hold:

Σ1. $\emptyset\in\Sigma_\Omega$,

Σ2. $A\in\Sigma_\Omega \Rightarrow (\Omega\setminus A)\in\Sigma_\Omega$, and

Σ3. for a collection of sets $A_i\in\Sigma_\Omega$, where index i takes values in a countable set D, follows that $(\bigcup_{i\in D} A_i) \in \Sigma_\Omega$.

In other words, a σ-algebra includes the empty set and it is closed under unions and countable intersections of its sets.

A **measure** is defined as a real, non-negative function $m_{\Sigma_\Omega}: \Sigma_\Omega \to R_0^+$ such that the following hold:

M1. $m_{\Sigma_\Omega}(\emptyset)=0$, and

M2. for every countable set of indices D, and for collection of disjoin subsets $A_i\in S$ it is valid that $m_{\Sigma_\Omega}(\bigcup_{i\in D} A_i)=\sum_{i\in D} m_{\Sigma_\Omega}(A_i)$.

The pair (Ω,Σ_Ω) is called **measurable space**. The exact definition of measure space $(\Omega,\Sigma_\Omega,m_{\Sigma_\Omega})$, is provided in the Appendix of this Chapter. A measure space offers the ability to handle non-countable data, that are elements of a set Ω. In particular, given a measure space $(\Omega,\Sigma_\Omega,m_{\Sigma_\Omega})$, the $(\Sigma_\Omega,\subseteq)$, where ⊆ is the common set-theoretical relation of subset, is a complete lattice with $o=\emptyset$ and $i=\Omega$. The operations of meet and join in lattice $(\Sigma_\Omega,\subseteq)$ are the set-theoretical operations of intersection (∩) and union (∪), respectively. The function of measure $m_{\Sigma_\Omega}(.)$is a positive valuation function in lattice $(\Sigma_\Omega,\subseteq)$. Finally, function θ(A)=

$\Omega\backslash A= A'$, that represents a set $A\in\Sigma_\Omega$ to its complement A', is a dual isomorphic function in lattice $(\Sigma_\Omega,\subseteq)$. Thus, all mathematical tools proposed in the previous sections are available in lattice $(\Sigma_\Omega,\subseteq)$.

A special case of measure space is the *probability space*, which is defined as a measure space $(\Omega, \Sigma_\Omega, m_{\Sigma_\Omega})$ with the constraint $m_{\Sigma_\Omega}(\Omega) = 1$. In some applications, the set Ω of a measure space $(\Omega, \Sigma_\Omega, m_{\Sigma_\Omega})$ is finite, and σ-algebra Σ_Ω is then equal to the powerset 2^Ω.

It is worth mentioning that an extension of the *powerset* concept, is the *fuzzy powerset* in relation to a reference set Ω, symbolically $\mathcal{F}(\Omega) = [0,1]^\Omega$. It has been proven that the structure $(\mathcal{F}(\Omega),\subseteq)$ is a complete lattice.

7.3.3 Statements

The set of (true/false) statements is a *Boolean algebra* (Birkhoff, 1967).

Assume the set of (true/false) statements Ω of interest on a particular application and assume that Σ_Ω is the powerset of set Ω. A *measure* can be defined, i.e. a function $m_{\Sigma_\Omega}: \Sigma_\Omega \rightarrow R_0^+$, by defining a positive number for every true statement. So, according to the above, the triple $(\Omega, \Sigma_\Omega, m_{\Sigma_\Omega})$ is a *measure space*. Therefore, all the mathematical tools proposed in the previous sections become available in the presumptive reasoning.

7.3.4 Trees

"Trees" are representations that can be used as mechanisms for decision making and/or studying processes where each tree node represents a decision/action (operation) that can be taken/executed, respectively. For example, Figure 7.2 illustrates a binary tree in which each node has exactly two "child nodes".

A tree represents a **join lattice**, where each pair of nodes x and y has a join $x\sqcup y$, but no meet $x\sqcap y$. In particular, the two-node join $x\sqcup y$ is the first node where the paths from x and from y to the root of the tree are encountered. For example, in Figure 7.2, it is $c_1\sqcup c_4 = c_{13}$, $c_5\sqcup c_{12} = c_{14}$, $c_2\sqcup c_{14} = c_{15}$, etc.

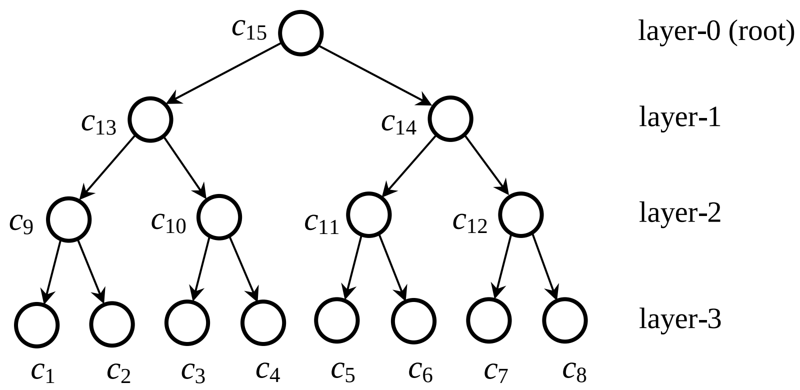

Fig. 7.2 *Binary tree with three layers and 2^3=8 leaves (nodes) in layer-3.*

In order to compare two tree nodes in Figure 7.2 using either a metric function or a fuzzy order function, a valuation process is applied, by giving positive values to tree leaves at level-3 as illustrated in Figure 7.3(a). Next, the valuation of the nodes of the exact previous level, level-2, takes place by summing the values of the children of each node as shown in Figure 7.3(b). More specifically, if c_i and c_j are children nodes of node c_k with values $v(c_i)$ and $v(c_j)$, respectively, then the parent node value c_k is calculated as $v(c_k)= v(c_i)+v(c_j)$. Progressively, nodes of all other levels are valuated up to the root of the tree as presented in Figure 7.3(c).

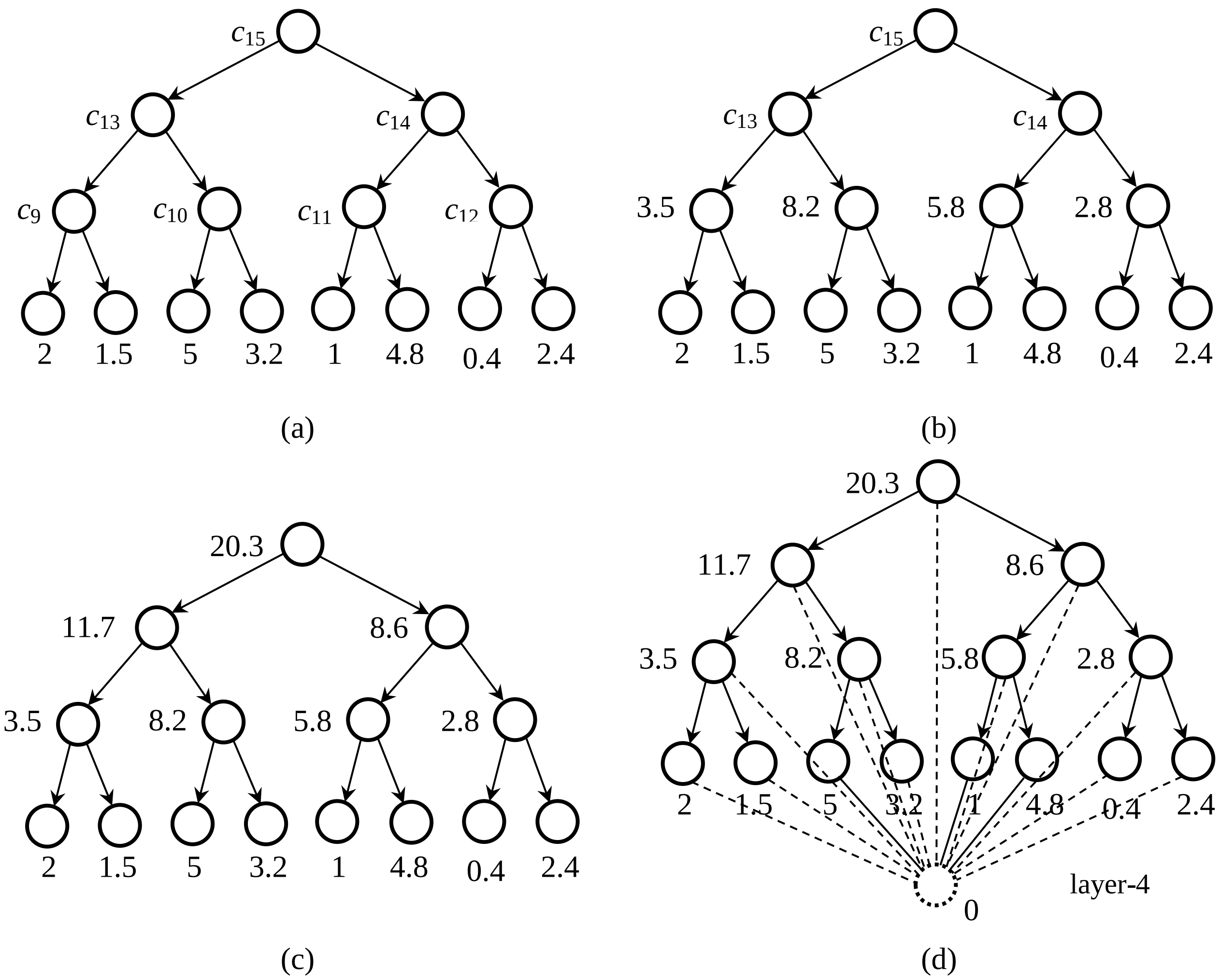

Fig. 7.3 *Valuation of all nodes of the binary tree of Fig. 7.2.*
(a) Valuation of the tree leaves.
(b) Valuation of tree nodes of level-2.
(c) The completed valuation of all tree nodes.
(d) The calculated values define a positive valuation function in the complete lattice derived after the import of a unique node (in level-4) as the least element (o) of the complete lattice.

The tree of Figure 7.3(c) can be transformed into a complete lattice by inserting an additional node in the (new) level-4 as the least element (o) with a value equal to 0 (see Figure 7.3(d)). Note that the greatest element (i) of the complete lattice in Figure 7.3(d) is the root-node of the tree c_{15}, i.e. $i= c_{15}$. Now, a metric function (d) and a fuzzy order (σ) can be calculated. For example, $d(c_1,c_{13})= v(c_1\sqcup c_{13})-v(c_1\sqcap c_{13})= v(c_{13})-v(c_1)= 11.7 – 2= 9.7$ and $d(c_3,c_{13})= v(c_3\sqcup c_{13})-v(c_3\sqcap c_{13})= v(c_{13})-v(c_3)= 11.7 – 5= 6.7$. Note that although nodes c_1 and c_3 are in the same level-3, node c_3 is calculated to be closer to node c_{13} rather than to node c_1 due to the used valuation function. Moreover, the distance between nodes c_1 and c_3 is calculated as $d(c_1,c_3)= v(c_1\sqcup c_3)-v(c_1\sqcap c_3)= v(c_{13})-v(\emptyset)= 11.7 - 0= 11.7$. That is, nodes c_1 and c_3 are more distant from each other than each one of them from node c_{13}. Following these, fuzzy orders are calculated.

For comparable elements, as for elements c_5 and c_{14}, results that $\sigma_\sqcup(c_5\sqsubseteq c_{14})= \frac{v(c_{14})}{v(c_5\sqcup c_{14})}= \frac{v(c_{14})}{v(c_{14})}= 1$, $\sigma_\sqcap(c_5\sqsubseteq c_{14})= \frac{v(c_5\sqcap c_{14})}{v(c_5)}= \frac{v(c_5)}{v(c_5)}= 1$. For non-comparable elements, such as elements c_1 and c_{14}, results that $\sigma_\sqcup(c_1\sqsubseteq c_{14})= \frac{v(c_{14})}{v(c_1\sqcup c_{14})}= \frac{v(c_{14})}{v(c_{15})}= \frac{8.6}{20.3}\cong 0.423$ and $\sigma_\sqcap(c_1\sqsubseteq c_{14})= \frac{v(c_1\sqcap c_{14})}{v(c_1)}= \frac{v(\emptyset)}{v(c_1)}= \frac{0}{2}= 0$. It is interesting to computationally verify that, for comparable elements, the degree to which a greater element is embedded in a smaller one may be non-zero. For example, $\sigma_\sqcup(c_{14}\sqsubseteq c_5)= \frac{v(c_5)}{v(c_{14}\sqcup c_5)}= \frac{v(c_5)}{v(c_{14})}= \frac{1}{8.6}\cong 0.116 \cong\frac{v(c_{14}\sqcap c_5)}{v(c_{14})}= \sigma_\sqcap(c_{14}\sqsubseteq c_5)$.

All previously mentioned techniques can be extended as described below.

First extension: To the aforementioned binary tree, if c_i and c_j are children nodes of c_k with values $v(c_i)$ and $v(c_j)$, respectively, then the parent value c_k can be put as $v(c_k) > v(c_i)+v(c_j)$. This results to the need for additional elements in lattice of Figure 7.3(d) so that for each pair c_i and c_j, a meet $c_i\sqcap c_j$ με $v(c_i\sqcap c_j) > 0$ exists. However, in order to calculate metric distances or/and fuzzy orders, the only thing needed is value $v(c_i\sqcap c_j)$ and not element $c_i\sqcap c_j$ itself. Note that value $v(c_i\sqcap c_j)$ must be calculated so as to fulfill the following two equations; $(v(x)+v(y) = v(x\sqcap y)+v(x\sqcup y)$ and $x \sqsubset y \Rightarrow v(x) < v(y)$, of a positive valuation function. Thus, for a given arithmetic value $v(c_i\sqcap c_j)$, are resulted $d(c_i,c_j)= 2v(c_i\sqcup c_j)-v(c_i)-v(c_j)$ and $\sigma_\sqcap(x,u) = \frac{v(x)+v(u)-v(x\sqcup u)}{v(x)}$.

Second extension: The tree might not be binary. In this case, a valuation process is applied by giving positive values to the tree leaves. Then, the nodes of the exact above level are valuated. Assuming that a node c_k has the children nodes c_i, $i\in I$. The value $v(c_k)$ is calculated by posing $v(c_k) > \max_{\substack{i,j\in I\\ i\neq j}}\{v(c_i)+v(c_j)\}$. Gradually, i.e. per level, all nodes are valuated up to the root of the tree.

In both extensions, the problem of valuating the nodes of a tree can be encountered as an optimization problem.

It should be noted that the calculation of the meet $c_i \sqcap c_j$ in a tree, where c_i and c_j are nodes, is necessary for calculating intervals (within trees). However, as already explained, the calculation of the meet $c_i \sqcap c_j$ is not apparent in a general tree structure except for special cases such as the binary tree of Figure 7.2, an extension of which in complete lattice is shown in Figure 7.3(d). Here, the optimization problem is more complex, since additionally to the valuation of nodes, new nodes have to be inserted.

The goal is to compute a lattice $(\mathsf{L},\sqsubseteq)$ that is equipped with both a positive valuation function $v(.)$ and a dual isomorphic function $\theta(.)$ so that the original tree of interest to be embedded in the lattice $(\mathsf{L},\sqsubseteq)$, i.e. to be a sublattice of $(\mathsf{L},\sqsubseteq)$. The aforementioned construction problem is similar to the one described in Section 9.1.2 where the lattice $(\mathsf{I}_1,\subseteq)$ is embedded in the generalized intervals lattice $(\mathsf{R}\times\mathsf{R},\geq\times\leq)$ where we can calculate both positive valuation functions $v(.)$ as well as dual isomorphic functions $\theta(.)$. However, the corresponding construction in trees is considered to be a more complex problem since a different lattice $(\mathsf{L},\sqsubseteq)$ must be constructed for each different tree.

7.3.5 Conclusion

In addition to the specific examples of lattices presented in this chapter, there are other useful representations for analysis in the context of order theory. For example, *ontologies* (Guarino, 2009) are a popular representation in computer science, e.g. in the *semantic web*, which could be embedded in a lattice, in a similar way as explained in Section 7.3.3 for trees, in order to result into the useful mathematical tools presented in this chapter.

The order theory is a field for the computation with semantics which are represented by partially ordered relation in a lattice. Generally, order theory allows for more effective representations. Moreover, due to the fact that the Cartesian product of (dissimilar) lattices is also a lattice, order theory arises as a strict mathematically framework for the unification as well as for a strict **disparate data fusion** in modeling applications in the CI. Finally, it should be noted that an algorithm applicable to lattices, e.g. a learning algorithm, has a wide application field without substantial modifications. Selected applications to support the aforementioned advantages of order theory are presented in Chapter 9.

APPENDIX of Chapter 7 (General Lattice Theory)

In the Appendix the fundamental concepts and the basic knowledge of lattice theory are presented.

Binary Relation "Partial order"

A **binary relation** R (between two sets P and Q) is defined as a subset of the Cartesian product $P \times Q$, i.e. $R \subseteq P \times Q$. Instead of $(p, q) \in R$ it can be equally written as pRq. If $P = Q$ then it is about a binary relation in (one) set. The **inverse** relation of R is symbolized as R^{-1}, i.e. it is $qR^{-1}p :\Leftrightarrow pRq$ by definition.

A binary relation $R \subseteq P \times Q$ is called **function** when there are no pairs $(p,q_1) \in R$ and $(p,q_2) \in R$ with $q_1 \neq q_2$. In other words, a function f is a correspondence that depicts every element p of set P in a single element $f(p)$ of set Q. The $f(p)$ is called **image** of p. If every element of set Q is the image of an element of set P then it is said that function f is **surjection**. In addition, if the inverse relation of a function is also a function then the function is called **bijection**. In other words, bijection means "1-1 and surjection". A specific binary relation is considered below.

A binary relation $R \subseteq P \times P$ in a set P is called **partially ordered** if and only if it satisfies the following:

Δ1. $(x,x) \in R$ (Reflexity)

Δ2. $(x,y) \in R$ and $x \neq y \Rightarrow (y,x) \notin R$ (Antisymmetry)

Δ3. $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$ (Transitivity)

Condition Δ2 can be replaced by the following equivalent condition:

Δ2′ $(x,y) \in R$ και $(y,x) \in R \Rightarrow x = y$ (Antisymmetry)

Instead of $xRy \Leftrightarrow (x,y) \in R$ is also used the representation $x \sqsubseteq y \Leftrightarrow (x,y) \in \sqsubseteq$ and it is said that “x is contained in y”, or “x is part of y”, or “x less than or equal to y”. If $x \sqsubseteq y$ and $x \neq y$, it is written that $x \sqsubset y$, and it is said that “x is strictly less than y” or “x is contained strictly in y”. In the same way are defined the symbols $x \sqsupseteq y$ και $x \sqsupset y$ for the inverse relation R^{-1}.

As a general information, it is worth to mention that the aforementioned definition of partially ordered relation differs from the definition of the (also binary) *equivalence relation* only in the (anti)symmetric condition, as it is shown below.

A binary relation $R \subseteq P \times P$ in a reference set P is called **equivalence relation** if and only if it satisfies the following:

I1. $(x,x) \in R$ (Reflexity)

I2. $(x,y) \in R \Rightarrow (y,x) \in R$ (Symmetry)

I3. $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$ (Transitivity)

Partially ordered set (poset) is a pair $(P,\sqsubseteq)$, where P is a set and $\sqsubseteq$ is a (partial) order relation in P. Note that in the same set P can be defined more than one (different) order relations, e.g. $\sqsubseteq_1$ and $\sqsubseteq_2$.

A function $\phi: P \rightarrow Q$ of the poset $(P,\sqsubseteq)$ to the poset $(Q,\sqsubseteq)$ is called **order preserving**, or **monotone**, if $x \sqsubseteq y \Rightarrow \phi(x) \sqsubseteq \phi(y)$, for $x, y \in P$. If, additionally, ϕ satisfies the inverse relation $x \sqsubseteq y \Leftarrow \phi(x) \sqsubseteq \phi(y)$, then ϕ is called **order embedded**. A *bijection function of embedded order* is called **isomorphism**. When an isomorphism function exists between two poset $(P,\sqsubseteq)$ and $(Q,\sqsubseteq)$, then the poset $(P,\sqsubseteq)$ and $(Q,\sqsubseteq)$ are called *isomorphic*, and are symbolized as $(P,\sqsubseteq) \approx (Q,\sqsubseteq)$. The **dual** of a poset $(P,\sqsubseteq)$ is a poset $(P,\sqsubseteq)^{\partial} = (P,\sqsubseteq^{\partial}) = (P,\sqsupseteq)$

defined from the inverse order relation on the same elements. If for two poset $(P,\sqsubseteq)$ and $(Q,\sqsubseteq)$ is implied that $(P,\sqsubseteq) \approx (Q,\sqsubseteq)^{\partial}$, then the $(P,\sqsubseteq)$ and $(Q,\sqsubseteq)$ are called **dually isomorphic**.

Duality principle (to a poset): the inverse $\sqsupseteq$ of an order relation $\sqsubseteq$ is also an order relation. More specifically, the inverse order $\sqsupseteq$ is called **dual** of $\sqsubseteq$ and is symbolized as $\sqsubseteq^{\partial}$, ή $\sqsubseteq^{-1}$, or $\sqsupseteq$. The dual principle is used to extend definitions and proofs, as explained below.

The **dual statement** S^{∂} of a statement S on a poset is received, if the symbols $\sqsubseteq$ and $\sqsupseteq$ are mutually exchanged in statement S. S is valid for an ordered set if and only if S^{∂} is valid for its own **dual** set.

Two different elements x and y of a poset are called **comparable** if $x \sqsubseteq y$ or $y \sqsubseteq x$. **Incomparable** elements x and y are also called **parallel** and are symbolized as $x||y$.

A special case of a poset is presented below. **Chain** or **totally ordered set** is a poset $(P,\sqsubseteq)$ which contains only comparable elements. For example, set R of real numbers with the common order relation $\leq$ is a poset, and particularly poset $(R, \leq)$ is a chain. An example of a poset that is not a chain is presented below, referring to Figure 7.4.

A finite poset $(P,\sqsubseteq)$ can be represented with a **Hasse diagram**, where the elements of P are depicted with small circles (nodes) so as the two elements $a \in P$ and $b \in P$, respectively, "up" and "down" to the diagram, to connect with one line if and only if a covers b – By saying that "a **covers** b" in a poset $(P,\sqsubseteq)$ it means that $a \sqsupset$ b and, additionally, does not exist $x \in P$ such that $a \sqsupset x \sqsupset b$. Figure 7.4 demonstrates the Hasse diagram of **powerset** $2^{\{a,b,c\}}$, where $\{a,b,c\}$ covers the $\{a,b\}$, $\{a,c\}$ covers $\{c\}$, etc. – As a **powerset** of a set S, symbolized as 2^S, is defined the total of all subsets of S.

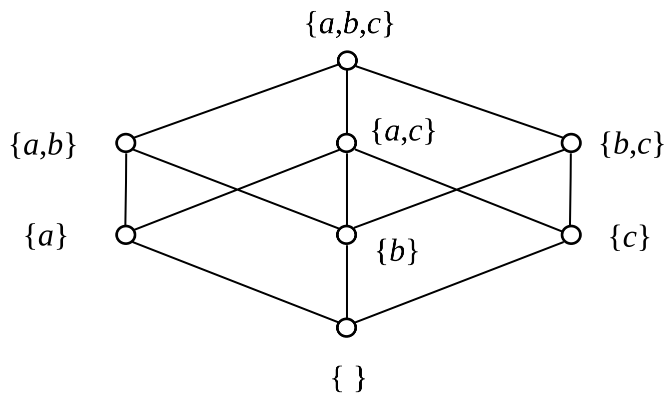

Fig. 7.4 *Hasse diagram of powerset* $2^{\{a,b,c\}}$ *of set* $S = \{a, b, c\}$.

The **least** element, if exists, of a set $X \subseteq P$ to a poset $(P,\sqsubseteq)$ is the unique element $a \in X$ such that $a \sqsubseteq x$ for every $x \in X$. The corresponding dual element in a set $X \subseteq P$ is called **greatest** element. The least element, if exists, in a poset is symbolized with *o*, while the greatest element is symbolized with *i*.

Given a poset $(P,\sqsubseteq)$ with least element *o*. Every $x \in P$ that covers *o*, if such x exists, is called **atom**. E.g., sets $\{a\}$, $\{b\}$, $\{c\}$ in Figure 7.4 are atoms.

Given a poset $(P,\sqsubseteq)$ and $a, b \in P$ with $a \sqsubseteq b$. As **ordinary interval** $[a, b]$ is defined the set $[a, b]:= \{x \in P: a \sqsubseteq x \sqsubseteq b\}$.

Assume that I is the set of ordinary interval in $(P,\sqsubseteq)$. The poset $(\mathrm{I},\subseteq)$ is resulted, where $\subseteq$ is the subset relation. The order relation $[a,b]\subseteq[c,e]$ is equivalent to the relation $c\sqsubseteq a \sqsubseteq b\sqsubseteq e$. The poset $(\mathrm{I},\subseteq)$ can be extended by inserting a least element which is the empty set $(\emptyset)$. Thus, it results the poset $(\mathrm{I}\cup\{\emptyset\},\subseteq)$. Note that the aforementioned equivalent order relation does not extend in poset $(\mathrm{I}\cup\{\emptyset\},\subseteq)$ since an obvious representation of the empty set $(\emptyset)$ in form of interval, does not exist. Every trivial interval $[x,x]\in\mathrm{I}$ in a poset $(\mathrm{I}\cup\{\emptyset\},\subseteq)$ is an atom.

If $(P,\sqsubseteq)$ is a poset, the set $(a] := \{x \in P: x \leq a\}$ is called **principal ideal** (derived from a), while set $[b) := \{x \in P: x \geq b\}$ is called **principal filter** (derived from b).

Size of an element of a poset $(P,\sqsubseteq)$ is defined a (non-negative) real function $\delta: P\rightarrow R_0^+$ that satisfies the following:

S1. $u \sqsubset w \Rightarrow \delta(u) < \delta(w)$.

The *Cartesian product* N of a poset $(P_1,\sqsubseteq_1)\times\ldots\times(P_N,\sqsubseteq_N)$ is defined as a poset $(P_1\times\ldots\times P_N,\sqsubseteq) = (P_1\times\ldots\times P_N,\sqsubseteq_1\times\ldots\times\sqsubseteq_N)$ με $(x_1,\ldots,x_N)\sqsubseteq(y_1,\ldots,y_N) :\Leftrightarrow x_1\sqsubseteq_1 y_1, \ldots, x_N\sqsubseteq_N y_N$.

Assume that the poset $(P,\sqsubseteq)$ is equal to the Cartesian product $(P, \sqsubseteq) = \times_{i\in\Omega} (P_i, \sqsubseteq_i)$, where Ω is a totally ordered set of elements, assume a function of *size* δ_i: $P_i\rightarrow R_0^+$ in every poset $(P_i,\sqsubseteq_i)$, and assume the *probability space* $(\Omega, \Sigma_\Omega, P_{\Sigma_\Omega})$. Then, the *size* of an element $A\in(P,\sqsubseteq)$, with components $A_i\in P_i$, $i\in\Omega$, is a function δ: $P\rightarrow R_0^+$, that is calculated from the general equation:

$$\delta(A) = \int_\Omega \delta_i(A_i)dP_{\Sigma_\Omega}.$$

Note that Ω can be either discrete or continuous. In all cases, the function of probability measure $P_{\Sigma_\Omega}: \Sigma_\Omega \rightarrow [0,1]$ is interpreted as *weight function*.

The Binary Relation "Lattice"

Lattice theory, or **order theory** was introduced by Garrett Birkhoff (Birkhoff, 1967; Davey & Priestley, 1990; Grätzer, 2003). In the remaining of the chapter, useful elements of this theory are summarized.

Assume a poset $(P,\sqsubseteq)$ and $X \subseteq P$. **Upper bound** of X is an element $a\in P$ with $x \sqsubseteq a$, $\forall x\in X$. **Least upper bound** of X, if it exists, is the unique upper bound contained in every upper bound. The least upper bound is called **lattice join**, or simply **join**, of X and it is symbolized with $supX$ or $\sqcup X$. The concepts of the **lower bound** of X and the **greatest lower bound** of X are defined dually. The greatest lower bound is called **lattice meet**, or simply **meet**, of X and it is symbolized with $infX$ or $\sqcap X$. If $X = \{x, y\}$ then $x\sqcup y$ will be written for $supX$ and $x\sqcap y$ for $infX$. The first definition of lattice follows.

Lattice $(\mathsf{L},\sqsubseteq)$ is defined a poset in which any two elements $x, y \in L$ have greatest lower bound, symbolized with x⊓y, and least upper bound, symbolized with x⊔y.

Every chain, including the chain of real numbers $(R, \leq)$, is a lattice. A general lattice $(\mathsf{L},\sqsubseteq)$ is called **complete** when every subset of X has a greatest lower bound and least upper bound in L. Setting $X = L$ results that a (non-empty) complete lattice contains a **least** and a **greatest** element, symbolized with o and i, respectively.

The previous definition for lattice is called **semantic lattice definition**. In addition, there is an equivalent, second definition, which is called **algebraic lattice definition** and is given below, based on the binary operations meet (⊓) and join (⊔) – Recall that **algebra** A is defined a pair $[S,F]$, where S is a non-empty set, and F is a well-defined set of **operations** f_a each of which depicts a power $S^{n(a)}$ of S, in S for a finite positive integer $n(a)$. The second definition of lattice follows.

Lattice $(\mathsf{L},\sqsubseteq)$ is defined an algebra $(\mathsf{L},\sqcap,\sqcup)$ with two binary operations ⊓ and ⊔ that satisfy the conditions L1 - L4, and vice versa.

L1. $x\sqcap x = x$, $x\sqcup x = x$ (Identitarian)

L2. $x\sqcap y = y\sqcap x$, $x\sqcup y = y\sqcup x$ (Commutative)

L3. $x\sqcap(y\sqcap z) = (x\sqcap y)\sqcap z$, $x\sqcup(y\sqcup z) = (x\sqcup y)\sqcup z$ (Associative)

L4. $x\sqcap(x\sqcup y) = x\sqcup(x\sqcap y) = x$ (Absorption)

The binary relation $x \sqsubseteq y$ is equivalent to the pair of operations

$x\sqcap y = x$ and $x\sqcup y = y$ (Consistency)

The binary operations of meet (⊓) and joint (⊔) have different properties. For example, operations ⊓ and ⊔ are *monotones*, i.e. $y\sqsubseteq z \Rightarrow x\sqcap y \sqsubseteq x\sqcap z$ and $x\sqcup y \sqsubseteq x\sqcup z$, $\forall x\in \mathsf{L}$. More specifically, in a complete lattice $(\mathsf{L},\sqsubseteq)$ it is $o\sqcap x = o$, $o\sqcup x = x$, $x\sqcap i = x$, and $x\sqcup i = i$,

$\forall x \in \mathsf{L}$. By definition, a lattice is called **atomic** when every element is equal to the join of atoms.

In CI applications, the algebraic lattice definition is useful mainly for calculations, while the semantic lattice definition is useful for decision-making.

From a strictly mathematical point of view, note that lattice theory is not as general as set theory because of the limitations in lattice definition. However, in practice, lattice theory is offered for analysis and design of a wide range of applications as outlined in Section 7.3.

The *dual*, symbolized $(\mathsf{L},\sqsubseteq)^{\partial} \equiv (\mathsf{L},\sqsubseteq^{\partial}) \equiv (\mathsf{L},\sqsupseteq)$, of a lattice $(\mathsf{L},\sqsubseteq)$ is also a lattice where join and meet operations mutually alternate. In other words, every sentence in lattice $(\mathsf{L},\sqsubseteq)$ is also valid in lattice $(\mathsf{L},\sqsupseteq)$ under the condition that operation $\sqcap$ replaces the operation $\sqcup$ and operation $\sqcup$ replaces the operation $\sqcap$. According to the above, the dual of a complete lattice, is a complete lattice.

Duality principle (in a complete lattice): the *dual sentence* of a (theoretical) sentence in a complete lattice results if the symbols $\sqsubseteq$, $\sqcup$, $\sqcap$, o, i are replaced with the symbols $\sqsupseteq$, $\sqcap$, $\sqcup$, i, o, respectively.

Sublattice $(\mathsf{S},\sqsubseteq)$ of a lattice $(\mathsf{L},\sqsubseteq)$ is defined a lattice with $\mathsf{S}\subseteq\mathsf{L}$. The $(\mathsf{L},\sqsubseteq)$ is called **superlattice** of $(\mathsf{S},\sqsubseteq)$. Moreover, the (sub)lattice $(\mathsf{S},\sqsubseteq)$ is embedded in the (super)lattice $(\mathsf{L},\sqsubseteq)$.

If $(a,b)\in\sqsubseteq$ in a lattice $(\mathsf{L},\sqsubseteq)$ then $([a,b],\sqsubseteq)$, where $[a,b]$ is the closed interval $[a,b]:= \{x\in\mathsf{L}: a\sqsubseteq x\sqsubseteq b\}$, is a sublattice. A sublattice $(\mathsf{S},\sqsubseteq)$ of lattice $(\mathsf{L},\sqsubseteq)$ is called **convex,** when $a,b\in\mathsf{S}$ implies $[a\sqcap b,a\sqcup b]\subseteq\mathsf{S}$.

A lattice is called **distributive** if and only if any of the following two "distributive identities" are valid, for every x, y, z.

L5. $x\sqcap(y\sqcup z) = (x\sqcap y)\sqcup(x\sqcap z)$ L5′. $x\sqcup(y\sqcap z) = (x\sqcup y)\sqcap(x\sqcup z)$

Complement, if exists, of an element x in a complete lattice $(\mathsf{L},\sqsubseteq)$ with least (greatest) element $o(i)$, is called another element $y\in\mathsf{L}$ such that $x\sqcap y = o$ and $x\sqcup y = i$. A lattice $(\mathsf{L},\sqsubseteq)$ is called **complemented** if all of its elements have a complement. A complemented distributive lattice is called **Boolean lattice**.

An example of Boolean lattice is the pair $(2^S,\subseteq)$, where 2^S is the powerset of a set S and «$\subseteq$» is the set-theoretical relation of subset. Meet and join in Boolean lattice $(2^S,\subseteq)$ is the set-theoretical meet ($\cap$) and the set-theoretical join ($\cup$), respectively.

Boolean algebra is defined as an algebra $(\mathsf{L},\sqcap,\sqcup,')$ with two binary operations $\sqcap$, $\sqcup$ and one operation that satisfies L1 - L8.

L6. $x\sqcap x' = o$, $x\sqcup x' = i$

L7. $(x')' = x$

L8. $(x \sqcap y)' = x' \sqcup y'$ $(x \sqcup y)' = x' \sqcap y'$

Given the fact that all statements are a Boolean algebra, as explained in Section 7.3.3, it follows that, in the context of Classical Logic, the properties/operations of a Boolean algebra extend to all statements. Further extensions to Fuzzy Logic and Fuzzy Reasoning are presented in Section 8.1.

Assume lattices $(\mathsf{L},\sqsubseteq)$ and $(\mathsf{M},\sqsubseteq)$. A function φ: $\mathsf{L}\to\mathsf{M}$ is called:

(a) **Joint morphism**, if $\varphi(x \sqcup y) = \varphi(x) \sqcup \varphi(y)$, $x, y \in L$.

(b) **Meet morphism**, if $\varphi(x \sqcap y) = \varphi(x) \sqcap \varphi(y)$, $x, y \in L$.

A function φ is called (**lattice**) **morphism** if φ is, at the same time, joint morphism and meet morphism. A bijection morphism is called (**lattice**) **isomorphism**.

The following function is of particular importance in the context of this book.

Valuation function in a lattice $(\mathsf{L},\sqsubseteq)$ is a real function v: $\mathsf{L}\to\mathsf{R}$ that satisfies $v(x)+v(y) = v(x \sqcap y)+v(x \sqcup y)$. A valuation function is called **monotone** if and only if $x \sqsubseteq y \Rightarrow v(x) \leq v(y)$, and **positive** if and only if $x \sqsubset y \Rightarrow v(x) < v(y)$.

Observe that a positive valuation function could be a *size* function. In particular, a positive valuation function, which receives non-negative values, is a size function.

The (positive) valuation functions usually refer to lattice theory, without being given any special significance. On the contrary, in this book, positive valuation functions are critical because they allow the definition, as explained below, of two useful functions for the (quantified) comparison of lattice elements. Specifically, these two useful functions are, first, the fuzzy order function and, secondly, a metric function. One of the functions is presented in Section 7.1, while the other is presented below.

A monotone valuation function v: $\mathsf{L}\to\mathsf{R}$ in a lattice $(\mathsf{L},\sqsubseteq)$ results to a *pseudo-metric* function $d: L \times L \to R_0^+$ given by the relation $d(x,y)=v(x \sqcup y)-v(x \sqcap y)$, $x, y \in L$. Moreover, if the valuation function $v(.)$is positive then function $d(x,y) = v(x \sqcup y)-v(x \sqcap y)$ is *metric* function – The definition of (pseudo)metric function is provided below.

Given a set X, **metric** is called a non-negative function $d: X \times X \to R_0^+$ if and only if it satisfies the following:

M1. $d(x,y) = 0 \Leftrightarrow x = y$ (Identification)

M2. $d(x,y) = d(y,x)$ (Symmetry)

M3. $d(x,z) \leq d(x,y) + d(y,z)$ (Triangular inequality)

If M1 is satisfied only from one direction such that $d(x,y) = 0$ for any $x \neq y$, while M2 and M3 are satisfied, then function $d(.,.)$ is called **pseudo-metric**.

A set X with a metric d is called **metric space**, symbolized as (X,d).

When in a lattice $(\mathsf{L},\sqsubseteq)$ with greater element i there is a positive valuation function $v(.)$, a **reasonable requirement** is $d(x,i) < +\infty$, for every $x \in \mathsf{L}$, i.e. it is required that the distance of any element $x \in \mathsf{L}$ from the greatest element i is finite. The aforementioned requirement implies $d(x,i) = v(x \sqcup i) - v(x \sqcap i) = v(i) - v(x) < +\infty \Rightarrow v(i) < +\infty$.

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CHAPTER 8: COMPUTATIONAL METHODOLOGIES IN LATTICES

Lattice theory has emerged as a "by-product" of an attempt to mathematically standardize the propositional logic of Aristotle. Specifically, in the first half of the 19th century, George Boole's attempt to standardize propositional logic resulted in the introduction of the *Boolean algebra*. The study of the Boolean algebraic axioms in the late 19th century led Peirce and Schröder into the concept of (mathematical) lattice; independently, Dedekind's research on **ideals of algebraic numbers** also led to the introduction of the (mathematical) lattice Grätzer, 2003). In the next decades results of lattice theory were published in a hostile climate that the mathematical circles of the time were maintaining.

With the systematic work of Birkhoff (1967) from the 1930s at Harvard University in Boston began the general development of lattice theory and, ultimately, its emergence in a distinct field of mathematics. Birkhoff showed that lattice theory unified uncorrelated mathematical fields, such as linear algebra, logic, probability theory, and so on. An important contribution to the foundation of lattice theory had several mathematicians and/or **logicians** such as: Jónsson, Kurosh. Malcev, Ore, von Neumann and Tarski (Rota, 1997).

Lattice computing today is a trend in the CI. This chapter outlines three scientific calculation **paradigms** in lattices that include #1. *Logic and Reasoning*, #2. *Standard Concept Analysis*, and #3. *Mathematical Morphology*. Specifically, methodologies #1 and #2 rely on the *semantic* lattice definition and use the *partially ordered* binary relation, while Methodology #3 is based on the *algebraic* lattice definition and uses the binary operations of *join* and *meet*. Note that the unifying nature of lattice theory in the CI has been certificated by several researchers (Bloch & Maitre, 1995; Maragos, 2005; Nachtegael & Kerre, 2001).

8.1 Logic and Reasoning

Lattices have been used in various logical studies (Birkhoff & von Neumann, 1936; Edmonds, 1980; Gaines, 1978; Halmos & Givant, 1998). Furthermore, an interesting generalization of the concept of *fuzzy set* is the concept **L-fuzzy set** (Goguen, 1967). In particular, the membership function of an L-fuzzy set represents the reference set in a (general) complete lattice instead of depicting it exclusively in the complete lattice of closed interval [0,1]. The aforementioned idea extends to both *logic* and *reasoning* with the truth function of a sentence to obtain values in a complete lattice for a more effective representation of the uncertainty. Finally, the resulted propositional logic is called *L-propositional logic* (Xu et al., 2003).

8.1.1 Fuzzy implications. Basic concepts, definitions

Recall that in classical (binary) logic, the implication truth values (⇒) verify the following truth table.

⇒	0	1
0	1	1
1	0	1

Table 8.1 *Truth table of classical logic (⇒)*

The above Table refers to the binary set $\{0,1\}$ as follows:

$$a \Rightarrow b \equiv \overline{a} \vee b.$$

The implication (⇒) can be interpreted as a binary operation with truth values in the set $\{0,1\}$. In fuzzy logic the set of truth values is extended in the interval $[0,1]$ and, thus, implication can be extended to a binary operation in the interval $[0,1]$. So, *fuzzy implication* is a representation:

$$I: [0,1] \times [0,1] \rightarrow [0,1],$$

that satisfies the above Table of classic implication "⇒", when the interval $[0,1]$ is limited to the binary set $\{0,1\}$.

The extension of classic implication $a \Rightarrow b \equiv \overline{a} \vee b$, in fuzzy logic, is

$$I_S(a,b) = S(n(a),b), \qquad \forall a,b \in [0,1] \tag{8.1}$$

where S, T and n, symbolize *t-conorm* (fuzzy disjunction), *t-norm* (fuzzy conjunction) and fuzzy negation, respectively. S and T are *binary* compared to n, i.e. they satisfy De Morgan's laws.

Moreover, type $a \Rightarrow b \equiv \overline{a} \vee b$,can be written in the two-members logic as follows

$$a \Rightarrow b \equiv max\{x \in \{0,1\} |\ a \wedge x \leq b\}, \qquad \forall a,b \in \{0,1\}$$

and

$$a \Rightarrow b \equiv \overline{a} \vee (a \wedge b), \forall a,b \in \{0,1\}$$

The corresponding extensions in fuzzy logic are:

$$I_R(a,b) = sup\{x \in [0,1] |\ \ T(a,x) \leq b\}, \forall a,b \in [0,1] \tag{8.2}$$

and

$$I_{QL}(a,b) = S\big(n(a), T(a,b)\big), \forall a,b \in [0,1] \tag{8.3}$$

The fuzzy implications resulting from (8.1) are called *S–implications*, those resulting from (8.2) are called *R-implications*, and those resulting from (8.3) are called *QL–implications*. Apart from the above three fuzzy implications that are the most popular in the bibliography, note that more fuzzy implications have been defined.

Generalizations of the properties of classical implication, lead to the following properties which are perceived as logical axioms (of fuzzy implications):

A1. $a \leq b \Rightarrow I(a,x) \geq I(b,x)$. That is, the truth value of fuzzy implications increases as the truth value of the hypothesis decreases.

A2. $a \leq b \Rightarrow I(x,a) \leq I(x,b)$. That is, the truth value of the fuzzy implications increases as the truth value of the conclusions increases.

A3. $I(0,a) = 1$. That is, the false statement implies anything.

A4. $I(1,b) = b$. Neutrality of the true statement.

A5. $I(a,a) = 1$. That is, the fuzzy implications are true when the truth values of the antecedent and the consequence are equal.

A6. $I(a, I(b,x)) = I(b, I(a,x))$. This is a generalization of the equivalence $a \Rightarrow (b \Rightarrow x)$ and $b \Rightarrow (a \Rightarrow x)$, which is applied in classical implication.
A7. $I(a,b) = 1$ if and only if $a \leq b$. That is, fuzzy implications are true if and only if the consequence is at least as true as the antecedent.
A8. $I(a,b) = I(n(b), n(a))$ for a fuzzy negation n. That is, fuzzy implications are equally true when the negations of antecedent and consequence alternate.
A9. Function *I* is continuous. This property ensures that small changes in the truth values of the antecedent or the consequence do not provoke great changes in the truth values of the fuzzy implications.

Axioms A1 – A9 are not independent of one another, e.g. A3 and A5 are derived from A7, but not vice versa. A fuzzy implication does not always satisfy all A1 - A9 axioms. However, when a fuzzy implication satisfies all of the A1 - A9, it also satisfies the following theorem:

Theorem *Smets* και *Magrez*. A function $I: [0,1] \times [0,1] \to [0,1]$ satisfies axioms *A1 - A9* of fuzzy implications, for a fuzzy negation n, if and only if there is a strictly increasing continuous function $f: [0,1] \to 0, \infty)$, with $f(0) = 0$. Then it holds that:
$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b)), \forall a, b \in [0,1]$ and
$n(a) = f^{-1}(f(1) - f(a)), \forall a \in [0,1]$.

Apart from the aforementioned definition of fuzzy implication, which is the most prevalent in the literature, an alternative definition follows.

Definition *Fodor* και *Roubens*. Fuzzy implication is a description:

$$I: [0,1] \times [0,1] \to [0,1],$$

that satisfies the logical table of classical implication, when $[0,1]$ is limited to $\{0,1\}$ and additionally $\forall a, b \in [0,1]$ satisfies the following:
(i) $a \leq b \Rightarrow I(a,x) \geq I(b,x)$,
(ii) $a \leq b \Rightarrow I(x,a) \leq I(x,b)$,
(iii) $I(0,a) = 1$,
(iv) $I(a,1) = 1$,
(v) $I(1,0) = 0$.

8.1.2 Lattice Implication Algebra

Assume a complete lattice $(L, \vee, \wedge, O, I)$, with O and I the least and greatest element, respectively, assume a dual isomorphism function, and assume a function $\to: L \times L \to L$. The heptad $\mathcal{L} = (L, \vee, \wedge, ', \to, O, I)$ is called **lattice implication algebra** if for every $x, y, z \in L$:

(I_1) $x \to (y \to z) = y \to (x \to z)$.
(I_2) $x \to x = I$.
(I_3) $x \to y = y' \to x'$.
(I_4) If $x \to y = y \to x = I$ then $x = y$.
(I_5) $(x \to y) \to y = (y \to x) \to x$.

(L_1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.
(L_2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

Subsequently, three examples of *lattice implication algebra* are presented.

Example 1
Assume $(L,\vee,\wedge,')$ a Boolean lattice. For every $x, y \in L$ define $x \to y = x' \vee y$. then, the quartet $(L,\vee,\wedge,' \to)$ is a *lattice implication algebra*.

Example 2
Assume $L = [0,1]$, with operations $\vee$, $\wedge$, $'$, and $\to$ defined for every $x, y, z \in L$ as follows: $x \vee y = max\{x, y\}$, $x \wedge y = min\{x, y\}$, $x' = 1 - x$ and $x \to y = min\{x, 1 - x + y\}$. Then, the heptad $([0,1],\vee,\wedge,',\to,0,1)$ is a *lattice implication algebra*, also known as algebra of Łukasiewicz in the interval [0, 1].

Example 3
Assume $L = \{a_i | i = 1,2,\cdots,n\}$, with operations $\vee$, $\wedge$, $'$, and $\to$ defined for every $1 \leq j, k \leq n$ as follows: $a_j \vee a_k = a_{max\{j,k\}}$, $a_j \wedge a_k = a_{min\{j,k\}}$, $(a_j)' = a_{n-j+1}$ and $a_j \to a_k = a_{min\{n-j+k,n\}}$. Then, the heptad $(L,\vee,\wedge,',\to,a_1,a_n)$ is a *lattice implication algebra*, which is a Łukasiewicz algebra in the finite chain $a_1,a_2,\ldots,a_n$.

Assume $(L,\vee,\wedge,',\to,O,I)$ a *lattice implication algebra*. Then, for every $x, y, z \in L$ the following properties are valid:

(1) If $I \to x = I$ then $x = I$.
(2) $I \to x = x, x \to O = x'$.
(3) $O \to x = I\ x \to I = I$.
(4) $(x \to y) \to y = x \vee y$.
(5) $x \leq y$ if and only if $x \to y = I$.
(6) $((x \to y) \to y) \to y = x \to y$.
(7) $(x \to y) \to x' = (y \to x) \to y'$.
(8) If $x \leq y$ then $x \to z \geq y \to z$ and $z \to x \leq z \to y$.
(9) $(x \to y) \to ((y \to z) \to (x \to z)) = I$.

In the context of L-propositional logic, particular emphasis has been given on the **resolution principle** for an **automated reasoning** (see an automated proofing process) in applications where doubt is represented with logical values within a general complete lattice.

8.2 Formal Concept Analysis

Formal context is defined a triad (*G*, *M*, *I*), which includes two sets *G* and *M* and a binary function *I* between *G* and *M*. The elements of *G* are called **objects** while the elements of *M* are called **attributes**. In order to symbolically express that an object *g* has a relation *I* with an attribute *m*, we write that *gIm* ή $(g,m)\in I$ and said that "object *g* has the attribute *m*».

For a set of objects $A\subseteq G$ it is defined:

$$A' = \{m \in M \mid gIm \text{ for all } g \in A\}$$

That is, A' is the set of all the common attributes of the elements of A.

Respectively, for a set of objects $B \subseteq M$ it is defined

$$B' = \{g \in G \mid gIm \text{ for all } m \in B\}$$

That is, B' is the set of all objects with attributes to B.

As **formal concept**, or just *concept*, in a triad (G, M, I) of formal context, is defined a pair (A,B) with $A \subseteq G$, $B \subseteq M$, $A'=B$ and $B'=A$.

It is proven that the set of formal concepts in a triad of formal contexts forms a complete lattice (Ganter & Wille, 1999).

Formal concept analysis (FCA) is a scientific methodology that usually studies tables of contexts, i.e. a questionnaire or a database, in order to compute a formal concept lattice. The following example is indicative of the FCA.

<u>*Example 4*</u>

For Table 8.2 of formal contexts can be calculated the 19 formal concepts presented in Figure 8.1.

		Attributes								
		a	b	c	d	e	f	g	h	i
1	Leech	×	×					×		
2	Bream	×	×					×	×	
3	Frog	×	×	×				×	×	
4	Dog	×		×				×	×	×
5	Rice	×	×		×		×			
6	Stubble	×	×	×	×		×			
7	Beans	×		×	×	×				
8	Maize	×		×	×		×			

Table 8.2 *Table of contexts regarding a number of leaving organisms, with the following attributes: a: "needs water to live", b: "lives in the water ", c: "lives on the ground", d: "it needs chlorophyll to feed", e: "two leaves of seeds", f: "one leave of seeds", g: "autonomous movement", h: "has limbs", i: "mamal".*

Note that several algorithms have been proposed for calculating formal concept lattices in FCA (Caro-Contreras & Mendez-Vazquez, 2013). Also, FCA extensions have been proposed in CI (Belohlavek, 2000). FCA applications are often suggested for **retrieval** of information in **databases** (Carpineto & Romano, 1996; Priss, 2000). In addition, applications in ontologies (Formica, 2006) are popular.

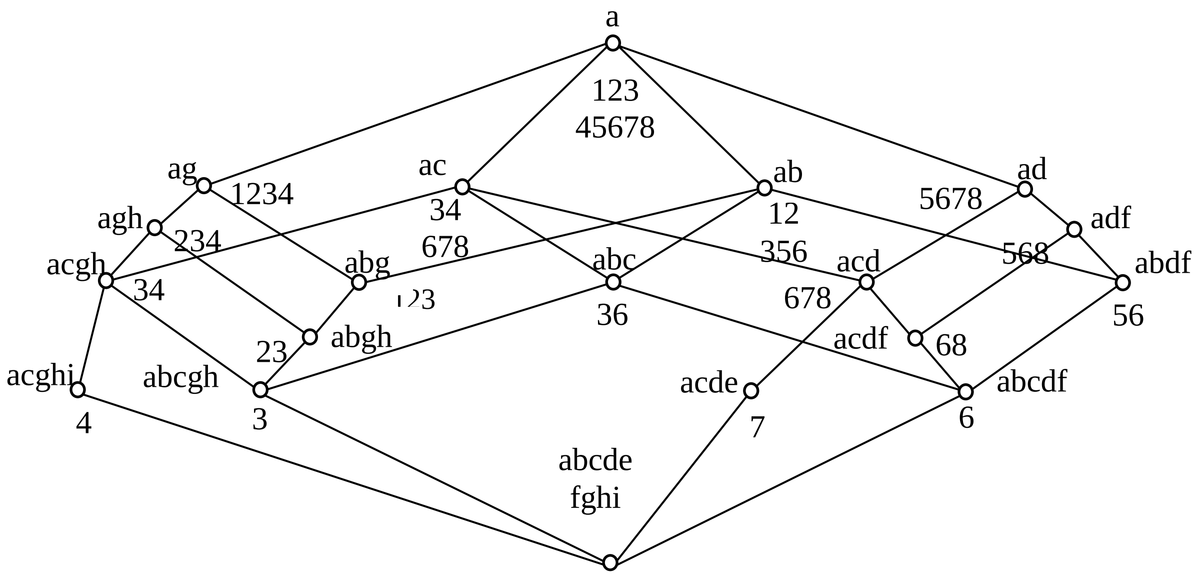

Figure 8.1 *Concept lattice calculated from Table 8.2.*

8.3 Mathematical Morphology

The **mathematical morphology (MM)** studies and designs techniques for the analysis and processing of geometric structures. It was proposed and founded by Matheron (1975) and Serra (1982), who developed a set of mathematical tools for image processing, considering images as sets of geometric forms and using extensively *lattice theory* for analysis.

Initially, MM was used to analyze binary images (sets of points) using set operations (Dougherty & Sinha, 1995). For the application of MM to grayscale images, set operations were generalized by the adoption of join, meet and encapsulation on the basis of lattice theory (Bloch et al., 2007). In particular, the application of MM techniques is typically modeled by using a two-dimensional **structure element** which scans a digital image by applying **dilation**, **erosion**, **opening** and **closing** operators in order to remove noise from the image and/or identify interesting **patterns** on the image. Note that given two complete lattices $(\mathsf{L},\sqsubseteq)$ and $(\mathsf{M},\sqsubseteq)$, *erosion* $\varepsilon : \mathsf{L}\rightarrow\mathsf{M}$ and *dilation* $\delta: \mathsf{L}\rightarrow\mathsf{M}$ are defines, respectively, as follows:

$$\varepsilon(\wedge M) =\wedge \varepsilon(M) \text{ και } \delta(\vee M) =\vee \delta(M),$$

where $\varepsilon(M)$ and $\delta(M)$ symbolize the sets $\{\varepsilon(\alpha)|\alpha\in M\}$ and $\{\delta(\alpha)|\alpha\in M\}$, respectively.

8.3.1 Morphological Operations in Image Processing

Assume $E = \mathbb{R}^n$ is non-empty set, 2^E the *powerset* of E and $\subseteq$ the binary operation of *encapsulation*. The pair $(2^E, \subseteq)$ is a **complete Boole lattice** (Meyer, 1991). An operation of sets is every depiction from $(2^E, \subseteq)$to itself. If $X, Y \in 2^E$ then operations $X \cup Y$, $X \cap Y$, $X\backslash Y$ and X^c are the usual total-theoretical operations of *join*, *meet*, *difference* and *complementarity*, respectively. Assume $h \in E$ and $X, B \subseteq E$, then set $X_h = \{x + h: x \in X\}$ is the **translation** of X by h, while set $X^t = \{-x: x \in X\}$ is the **inverse** of X.

Most morphological operations in sets result from the combination of the operations of sets, with the basic operations of *dilation* and *erosion* that result from Minkowski as follows:

$$X \oplus B = \cup_{b \in B} X_b \tag{8.4}$$
$$X \ominus B = \cup_{b \in B} X_{-b} \tag{8.5}$$

In morphological image processing applications, X corresponds to an image, B is the *structural element*, while the result of operations $X \oplus B$, $X \ominus B$ is transformed images. *Dilatation* and *erosion* operations, respectively, are defined using the building element $B \in P(E)$ as follows:

$$\delta_B = X \oplus B \tag{8.6}$$
$$\varepsilon_B = X \ominus B \tag{8.7}$$

Note that *dilation* and *erosion* are *dually complementary*. In particular, the dilation of a set equals to the erosion of the complement of the set, with structural element the inverse structural element as described in the following equations:

$$(X \oplus B)^c = X^c \ominus B^t \tag{8.8}$$
$$(X \ominus B)^c = X^c \oplus B^t \tag{8.9}$$

In practice, dilation inflates an object in the image, reduces the background, and deforms the convex angles of the object. Instead, erosion reduces the object, strengthens the background and distorts the concaved corners of the object.

The operations of dilation and erosion exist for every morphological operator, with most important, the operators of *opening* and *closing*. The last two operators are defined as follows:

$$X \circ \mathrm{B}=(X \ominus B) \oplus B \tag{8.10}$$
$$X \bullet \mathrm{B}=(X \oplus B) \ominus B \tag{8.11}$$

The opening operation removes the narrow parts of the object and deforms the convex angles of an object in the image, while the closing fills the narrow portions of the background and deforms the concave corners of an object in the image.

A key feature of *opening* and *closing* operations is that if they are applied repeatedly, they do not cause further changes after their first implementation. That is:

$$(X \circ B) \circ B = (X \circ B) \tag{8.12}$$
$$(X \bullet B) \bullet B = (X \bullet B) \tag{8.13}$$

8.3.2 Morphological Filters

The morphological operations of expansion, corrosion, opening and closing are applied in the form of filters, to remove noise from images, improve picture quality, extract features from images, etc. In particular, the *opening* can filter the *positive noise*, i.e. remove the noisy parts of the object, usually small portions. On the other hand, *closing* can be used to remove the *negative noise*, i.e. add noisy parts of the background to the object, usually small holes.

With the application of opening and closing, four new filters are resulted: (1) opening followed by closing, (2) closing followed by opening, (3) opening followed by closing and then opening, and (4) closing followed by opening and then closing.

In the literature, techniques for the design of morphological filters such as **coordinate logical filters** (Mertzios & Tsirikolias, 1998) and **amoeba filters** (Lerallut et al., 2007) have been proposed. In particular, the first category of filters is an alternative form of morphological filters that can be calculated quickly, while the second category is characterized by the flexible use of structural elements of variable size.

8.3.3 Extensions

MM techniques have been extended beyond image processing. For example, the *backpropagation* artificial neural network (ANN) Hopfield is presented as a MM technique.

Other MM studies attempt to replace the time-consuming operation of multiplication with the much faster act of meet between two numbers. In addition, according to the concept of *linearly independent* vectors, the concept of *independent lattice elements* has been studied in order to simplify the mathematical analysis in ANN applications, signal processing, etc. (Ritter & Gader, 2006; Ritter & Urcid, 2003; Ritter & Wilson, 2000).

8.4 Comparative Comments

Of the three methodologies for using mathematical lattices presented in this chapter, MM currently has the greatest relevance to CI. For example, several ANN have been proposed within MM (Pessoa & Maragos, 2000· Sussner & Graña, 2003· Yang & Maragos, 1995).

As noted at the beginning of this chapter, on one hand, both Logic/Reasoning and formal concept analysis rely on the *semantic* definition of a lattice and make use of the binary relation of *partial order*, while on the other hand, Mathematical Morphology is based on *algebraic* lattice definition and makes use mainly of the binary operations *join* and *meet*.

The use of the lattice theory proposed in Chapter 7 of this book emphasizes on the *semantic* definition of lattice. Furthermore, important differences in the use of lattice theory proposed in this book include the following: 1) lattices of non-numbered diversity can also be used here, and 2) the use of *positive valuation* functions here is critical.

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CHAPTER 9: INTERVAL NUMBERS (INs)

This chapter is based on the special theory of Chapter 7 (Lattice Theory) to present a popular lattice hierarchy which is developed gradually starting from the chain (R,≤) of real numbers. In addition, some innovative mathematical tools are presented giving new prospects to Computational Intelligence (CI). It is interesting to recall some facts regarding set R.

The set R of real numbers is derived as a result of **measurements** (Kaburlasos, 2006). In particular, a size of interest is defined by comparing it repeatedly with a (similar) standard size, called "measure", as well as subdivisions of the latter. Both the quotient and the remainder of a measurement define a real number.

The set R is under study for over approximately 2.500 years from the time of Pythagorean philosophers (6th century BC), who proclaimed that (natural) numbers are the essence of everything. In particular, Pythagoreans believed that harmony in universe is described by numbers, where any number could be represented as a fraction of two natural numbers. That is, the Pythagoreans considered only the numbers we now call **rational**.

A few centuries after its appearance, under the "burden" of the discovery of some non-rational (**irrational)** numbers, the Pythagorean School collapsed. Around the beginning of the 20th century it turned out that the set of irrational numbers is non-countable, i.e. irrational numbers are more than rational numbers. The set of all numbers (both rational and irrational) was named the set of real numbers, symbolically R. Various attributes of set R were studied; An interesting attribute, within this book, is that the set R of real numbers is totally ordered. In this chapter based on the total order of set R, a popular lattice hierarchy is presented.

9.1 A Popular Lattice Hierarchy

In this section a popular six-level lattice hierarchy is presented. An additional level is outlined as an extension of the above six levels.

9.1.1 Level-0: The Lattice(R,≤) of Real Numbers

Chain (R,≤) of real numbers is not a complete lattice (Davey & Priestley, 1990). However, it can be converted into a complete lattice by inserting a least element $o = -\infty$ and a greatest element $i = +\infty$, thereby results a complete lattice $(\bar{R} = R \cup \{-\infty, +\infty\}, \leq)$.

Assume a complete lattice (L=[o,i],≤) of real numbers, with least and greatest elements $o \in \bar{R}$ and $i \in \bar{R}$, respectively, where $o<i$. The *greatest lower bound* of two numbers x and y is the smallest of the two, symbolized as $x \wedge y$, while the *least upper*

bound of two numbers is the greatest of the two, symbolized as $x \vee y$. As *positive valuation function v*: $L \rightarrow R_0^+$ in lattice (L,≤) is assumed a strictly increasing function such that it satisfies two *reasonable constraints* $v(o)=0$ and $v(i)<+\infty$. Moreover, as *dual isomorphic function* θ: L→L in lattice (L,≤) is assumed a strictly decreasing function such that $\theta(o)=i$ and $\theta(i)=o$. For example, first, to the complete lattice (L=[-∞,+∞],≤) the sigmoid function $v(x) = \frac{1}{1+e^{-x}}$ and the linear function $\theta(x)=-x$ can be considered. Secondly, to the complete lattice (L=[0,1],≤) functions $v(x)=x$ and $\theta(x)=1-x$, can be considered. Generally, parametric functions $\theta(.)$ and $v(.)$ introduce adjustable non-linearities whose parameters can be optimally estimated with various techniques, e.g. with evolutionary calculation, etc.

In any case, given of a positive valuation function v: $L \rightarrow R_0^+$, results a metric function d: $L \times L \rightarrow R_0^+$, defined from the equation $d(x,y)= v(x \vee y) - v(x \wedge y)$.

9.1.2 Layer-1: The Lattice (I_1,⊑) of Intervals Type-1 (T1)

Computation with intervals has a long history on handling uncertainties in calculations (Alefeld & Herzberger, 1983; Moore, 1979; Tanaka & Lee 1998). This chapter describes a different approach, in the context of lattice theory, with emphasis on semantics and (common) logic rather than algebra.

Consider a partially ordered lattice (I_1,⊑) of intervals T1 in a complete lattice (L=[o,i],≤) of real numbers. The *greatest lower bound* of two intervals T1 $[a,b]$ and $[c,e]$ is given from $[a,b]\sqcap[c,e]= [a\vee c,b\wedge e]$, if $a\vee c\leq b\wedge e$, and $[a,b]\sqcap[c,e]= \emptyset= [i,o]$, if $a\vee c>b\wedge e$. While, the *least upper bound* of two intervals T1 $[a,b]$ and $[c,e]$ is given by $[a,b]\sqcup[c,e]= [a\wedge c,b\vee e]$. Recall that the empty set ∅ in lattice (L=[o,i],≤) is symbolized as $[i,o]$.

Given of (a) a positive valuation function v: $L \rightarrow R_0^+$ and (b) a dual isomorphic function θ: L→L in lattice (L,≤), as already presented in Level-0, results a positive valuation function v_1: $L\times L \rightarrow R_0^+$ in lattice (L×L,≥×≤) of *generalized intervals*, given by the equation $v_1([a,b])= v(\theta(a))+v(b)$. Therefore, both a metric function $d_1(.,.)$ and two fuzzy order functions, $\sigma_\sqcap(.,.)$ and $\sigma_\sqcup(.,.)$, can be defined in lattice (L×L,≥×≤). The aforementioned functions are valid to the sublattice (I_1,⊑), which is embedded to the superlattice (L×L,≥×≤). In particular, the following three functions are available in (I_1,⊑).

A *metric* function d_1: $I_1\times I_1 \rightarrow R_0^+$ in lattice (I_1,⊑) is calculated as follows:

$$d_1([a,b],[c,e])= [v(\theta(a\wedge c))-v(\theta(a\vee c))] + [v(b\vee e)-v(b\wedge e)] = d(\theta(a),\theta(c)) + d(b,e) \quad (9.1)$$

Two fuzzy order functions $\sigma_\sqcap$: $I_1\times I_1\rightarrow[0,1]$ and $\sigma_\sqcup$: $I_1\times I_1\rightarrow[0,1]$ in lattice (I_1,⊑), are calculated as follows:

$$\sigma_{\sqcap}(x=[a,b],y=[c,e])=\begin{cases}1, & x=\emptyset\\ \frac{v_1(x\sqcap y)}{v_1(x)}=\frac{v(\theta(a\vee c))+v(b\wedge e)}{v(\theta(a))+v(b)}, & x\supset\emptyset\end{cases} \quad (9.2)$$

$$\sigma_{\sqcup}(x=[a,b],y=[c,e])=\begin{cases}1, & x\sqcup y=\emptyset\\ \frac{v_1(y)}{v_1(x\sqcup y)}=\frac{v(\theta(c))+v(e)}{v(\theta(a\wedge c))+v(b\vee e)}, & x\sqcup y\supset\emptyset\end{cases} \quad (9.3)$$

An interval T1 apart of the empty set, here it will be called *common interval T1*. The partially ordered set (poset) of all common intervals T1 will be denoted as $(I_{1p},\subseteq)$. The function δ_1: $I_{1p}\rightarrow R_0^+$, which is calculated as $\delta_1([a,b])= v_1([a,b])= v(\theta(a))+v(b)$ is *size* function is poset $(I_{1p},\subseteq)$. In particular, the size function $\delta_1([a,b])= v(b)-v(a)$ (referred in Chapter 7) is resulted as follows.

Functions $\theta(.)$ and $v(.)$ can be selected with various ways. For example, by selecting $\theta(x)= -x$ and $v(.)$ such that $v(x)=-v(-x)$, results the positive valuation function $v_1([a,b]) = v(b)-v(a) = \delta_1([a,b])$. Therefore, results the metric $d_1([a,b],[c,e])= [v(a\vee c)-v(a\wedge c)] + [v(b\vee e)-v(b\wedge e)]$. More specifically, for $v(x)= x$ results the L_1 metric (Hamming) $d_1([a,b],[c,e])= |a-c| + |b-e|$.

9.1.3 Layer-2: The Lattice $(I_2,\sqsubseteq)$ of Intervals Type-2 (T2)

A **type-2** (**T2**) **interval** is defined as an interval of intervals T1. For example, an interval T2 is the $[[a_1,a_2],[b_1,b_2]]$, where $[a_1,a_2]$ and $[b_1,b_2]$ are intervals T1, i.e.. $[a_1,a_2]$, $[b_1,b_2]\in(I_1,\sqsubseteq)$, with $[a_1,a_2]\sqsubseteq[b_1,b_2]$.

Assume $(I_2,\sqsubseteq)$ the partially ordered lattice of intervals T2 derived on a specific lattice $(I_1,\sqsubseteq)$ of intervals T1. The *greatest lower bound* of two intervals T2, $[[a_1,a_2],[b_1,b_2]]$ and $[[c_1,c_2],[e_1,e_2]]$, is given by the equations $[[a_1,a_2],[b_1,b_2]]\sqcap [[c_1,c_2],[e_1,e_2]]=[[a_1\wedge c_1,a_2\vee c_2],[b_1\vee e_1,b_2\wedge e_2]]$ if $[a_1\wedge c_1,a_2\vee c_2]\sqsubseteq[b_1\vee e_1,b_2\wedge e_2]$, and $[[a_1,a_2],[b_1,b_2]]\sqcap [[c_1,c_2],[e_1,e_2]]= \emptyset= [[o,i],[i,o]]$ if $[a_1\wedge c_1,a_2\vee c_2]\not\sqsubseteq [b_1\vee e_1,b_2\wedge e_2]$. While, the corresponding *lower upper bound* is given by the equation $[[a_1,a_2],[b_1,b_2]]\sqcup [[c_1,c_2],[e_1,e_2]]= [[a_1\vee c_1,a_2\wedge c_2],[b_1\wedge e_1,b_2\vee e_2]]$. Recall that the empty set $\emptyset$ in lattice $(I_2,\sqsubseteq)$ is symbolized as $[[o,i],[i,o]]$.

From Layer-1 recall the positive valuation function v_1: $L\times L\rightarrow R_0^+$ in lattice $(L\times L,\geq\times\leq)$, given by $v_1([a,b])= v(\theta(a))+v(b)$. Moreover, function θ_1: $L\times L\rightarrow L\times L$, given by the equation $\theta_1([a,b])= [b,a]$, is of *dual isomorphism* in lattice $(L\times L,\geq\times\leq)$ of generalized intervals. According to the above, results a positive valuation function v_2: $L\times L\times L\times L\rightarrow R_0^+$ in the complete lattice $(L\times L\times L\times L,\leq\times\geq\times\geq\times\leq)$ of generalized intervals, which is calculated by the equation $v_2([[a_1,a_2],[b_1,b_2]])= v_1(\theta_1([a_1,a_2]))+v_1([b_1,b_2])= v(a_1)+v(\theta(a_2))+v(\theta(b_1))+v(b_2)$. Thus, there can be defined both a metric function $d_2(.,.)$ and two fuzzy order functions $\sigma_{\sqcap}(.,.)$ and $\sigma_{\sqcup}(.,.)$ in lattice $(L\times L\times L\times L,\leq\times\geq\times\geq\times\leq)$. The aforementioned functions are valid in the sublattice $(I_2,\sqsubseteq)$, which is embedded in the superlattice (L×L×L×L, ≤×≥×≥×≤). More specifically, the following three functions are available in (I_2,⊑).

A *metric* function d_2: $I_2 \times I_2 \rightarrow R_0^+$ in lattice (I_2,⊑) is calculated as follows:

$$d_2([[a_1,a_2],[b_1,b_2]],[[c_1,c_2],[e_1,e_2]])= d(a_1,c_1)+d(\theta(a_2),\theta(c_2))+d(\theta(b_1),\theta(e_1))+d(b_2,e_2) \quad (9.4)$$

Two fuzzy order functions $\sigma_\sqcap$: $I_2 \times I_2 \rightarrow [0,1]$ and $\sigma_\sqcup$: $I_2 \times I_2 \rightarrow [0,1]$ in lattice (I_2,⊑), are calculated as follows:

$$\sigma_\sqcap([[a_1,a_2],[b_1,b_2]],[[c_1,c_2],[e_1,e_2]]) = \begin{cases} 1, & b_1 > b_2 \\ 0, & b_1 \le b_2, b_1 \vee d_1 > b_2 \wedge d_2 \\ 0, & b_1 \le b_2, b_1 \vee d_1 \le b_2 \wedge d_2, [a_1 \wedge c_1, a_2 \vee c_2] \nsubseteq [b_1 \vee e_1, b_2 \wedge e_2] \\ \frac{v_2([[a_1,a_2],[b_1,b_2]] \sqcap [[c_1,c_2],[e_1,e_2]])}{v_2([[a_1,a_2],[b_1,b_2]])}, & otherwise \end{cases} \quad (9.5)$$

$$\sigma_\sqcup\big([[a_1,a_2],[b_1,b_2]],[[c_1,c_2],[e_1,e_2]]\big) = \begin{cases} 1, & b_1 > b_2 \\ 0, & b_1 \le b_2, e_1 > e_2 \\ \frac{v_2([[c_1,c_2],[e_1,e_2]])}{v_2([[a_1,a_2],[b_1,b_2]] \sqcup [[c_1,c_2],[e_1,e_2]])}, & \text{otherwise} \end{cases} \quad (9.6)$$

An interval T2 apart of the empty set, here, it will be called **common interval T2**. The poset of all common intervals T2 will be symbolized as (I_{2p},⊆).

The *size* of a common interval T2, assuming $[[a_1,a_2],[b_1,b_2]]$, is a function δ_2: $I_{2p} \rightarrow R_0^+$, which is calculated as $\delta_2([[a_1,a_2],[b_1,b_2]])=v_1([b_1,b_2])-v_1([a_1,a_2])=v(\theta(b_1))+v(b_2)-v(\theta(a_1))-v(a_2)$.

9.1.4 Layer-3: The Lattice (F_1,≼) of Intervals' Numbers Type-1 (IN T1)

The *resolution identity theory* says that a fuzzy set can be equivalently represented either by its membership function or by its set of α-cuts. The resolution identity theory is used here as follows. In the beginning, the interpretation of the *feasibility* of a (fuzzy) membership function of fuzzy numbers is abandoned. Then, the corresponding representation with α-cuts is considered. Finally, an intervals' number (IN) is resulted, as explained in detail below. First, however, a more general type of number is defined.

A **Generalized Intervals' Number** (**GIN**) is defined as a function *f*: $[0,1] \rightarrow (\bar{R} \times \bar{R}, \ge \times \le)$, where $(\bar{R} \times \bar{R}, \ge \times \le)$ is a lattice of *generalized intervals*.

Assume G the set of GINs. Then, the (G,⊑) is a complete lattice, as a (non-countable) Cartesian product of complete lattices ($\bar{R}\times\bar{R}$,≥×≤). Then, the interest is focused to the sublattice of INs.

An **Intervals' Number Type-1** (**IN T1**), is defined as a function F: [0,1]→I_1, which satisfies the following two 1) $h_1 \leq h_2 \Rightarrow F_{h_1} \sqsupseteq F_{h_2}$ and 2) $\forall X \subseteq [0,1]$: $\bigcap_{h\in X} F_h = F_{\vee X}$.

The set F_1 of IN T1 is a partially ordered, complete lattice, which is denoted as (F_1,≼). An IN is interpreted as a *grain of information* (Kaburlasos & Papadakis, 2006). The set F_1 of INs has been studied in a series of studies. In particular, it has been shown that the set F_1 is a metric lattice (Kaburlasos, 2004) with *cardinality* equal to the infinity $\aleph_1$ of set R of the real numbers (Kaburlasos & Kehagias, 2006). In other words, there are as many INs, as real numbers. In addition, the set F_1 is a cone in a linear space (Papadakis & Kaburlasos, 2010).

An IN can be, equivalently, represented as a set of intervals F_h, $h\in[0,1]$; this is the **interval-representation,** or as a function $F(x)= \bigvee_{h\in[0,1]} \{h: x \in F_h\}$, this is the **membership-function-representation** as shown is Figure 9.1. The meet (⋏) and joint (⋎) in lattice (F_1,≼) are defined as $(F\curlyvee G)_h = F_h \sqcup G_h$ and $(F\curlywedge G)_h = F_h \sqcap G_h$, respectively. For example, Figure 9.2 presents the calculation of joint (⋎) and meet (⋏) of two INs F and G, respectively, by using membership-function-representations.

As *support* or *carrier* of an IN is defined the least upper bound interval $\bigvee_{h\in(0,1]} F_h$. Typically, the IN used in practice have continuous membership functions, thus, the equality $\bigvee_{h\in(0,1]} F_h = F_0$ is true, i.e. the support of a $F\in F_1$ is its "base" for h=0.

For INs $F,G\in F_1$ the following inequality has been proven (Kaburlasos & Kehagias, 2014):

$$F \preccurlyeq G \Leftrightarrow (\forall h\in[0,1]: F_h \sqsubseteq G_h) \Leftrightarrow (\forall x\in L: F(x) \leq G(x))$$

As *hight E* of an IN, symbolically $hgt(E)$, is defined the least upper bound (∨) of all corresponding membership degrees, i.e. $hgt(E) = \bigvee_{x\in[o,i]} E(x)$. For example, in Figure 9.1(a) it is $hgt(E)$= 1, while in Figure 9.2(c) it is $hgt(F\curlywedge G)= h_1$.

Subsequently, two fuzzy orders $\sigma_\curlywedge$: $F_1\times F_1\to[0,1]$ and $\sigma_\curlyvee$: $F_1\times F_1\to[0,1]$ are defined, in the base of fuzzy orders $\sigma_\sqcap$: $I_1\times I_1\to[0,1]$ and $\sigma_\sqcup$: $I_1\times I_1\to[0,1]$, respectively.

$$\sigma_\curlywedge(E,F)= \int_0^1 \sigma_\sqcap(E_h, F_h)\, dh \tag{9.7}$$

$$\sigma_\curlyvee(E,F)= \int_0^1 \sigma_\sqcup(E_h, F_h)\, dh \tag{9.8}$$

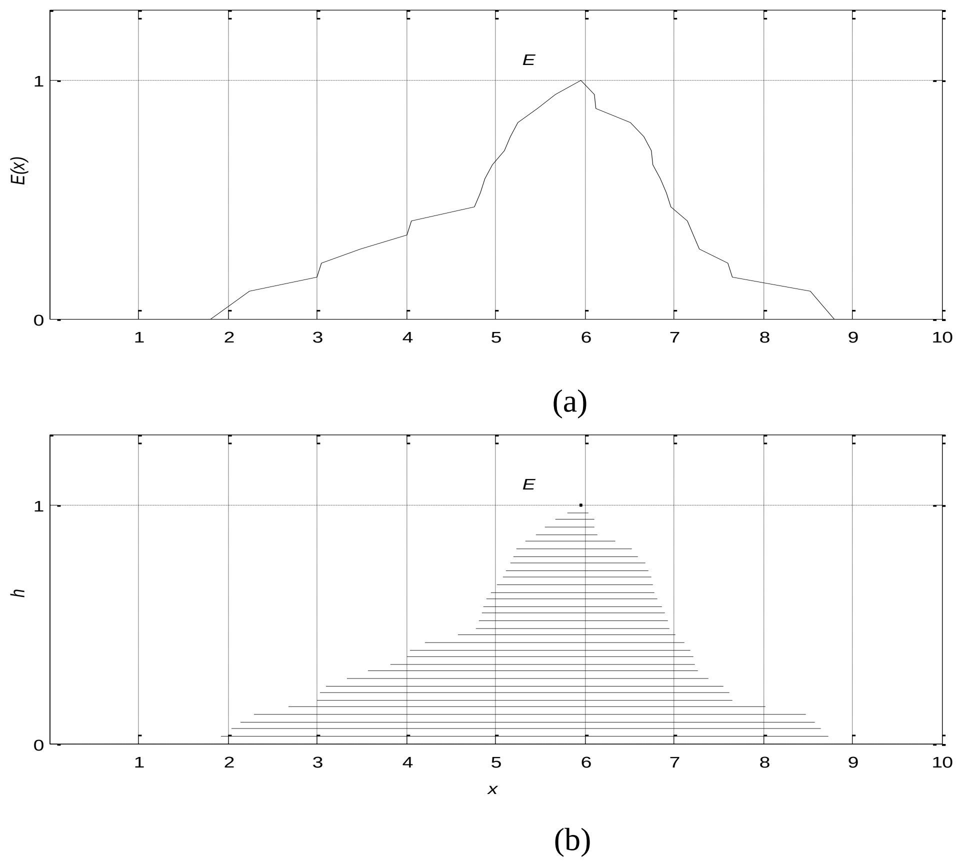


Figure 9.1 *The two equivalent representations of an Intervals' Number (IN): (a) the membership-function-representations and (b) the interval-representation.*

A (mathematical) result with important extensions is presented below, regarding Figure 9.3 (Kaburlasos & Kehagias, 2014):

$$F(x=t) = \sigma_\wedge(T,F) = \int_0^1 \sigma_\sqcap(T_h, F_h)\, dh, \tag{9.9}$$

where $F \in \mathrm{F}_1$ and $T= [t,t]$, $\forall h \in [0,1]$ is a *common* IN, which represents a real number. Specifically, Eq. (9.9) associates the two different, but equivalent, representations of a IN which are the membership-function-representation and the interval-representation via fuzzy order $\sigma_\wedge$: $\mathrm{F}_1 \times \mathrm{F}_1 \rightarrow [0,1]$ in the case that the first argument of the function $\sigma_\wedge(.,.)$ is a common IN $T= [t,t]$, $\forall h \in [0,1]$, while the second argument of the function $\sigma_\wedge(.,.)$ is any IN $F \in \mathrm{F}_1$ with membership function $F(x)$. Then, the fuzzy order $\sigma_\wedge(T,F)$ is equal to the value of function $F(x)$ for $x=t$. Moreover, the use of a fuzzy order function implies three basic advantages, as explained below.

First, the use of either function $\sigma_{\curlywedge}(T,F)$ of function $\sigma_{\curlyvee}(T,F)$ implies the ability to use non-common IN T for the representation of the *uncertainty/doubt*, secondly, the use of fuzzy order function $\sigma_{\curlyvee}(T,F)$ has the additional advantage that it is non-zero beyond the support F_0 of IN F and, thirdly, both functions $\sigma_{\curlywedge}(.,.)$ and $\sigma_{\curlyvee}(.,.)$ are parametric, therefore they can be optimized with estimation of their parameters. To all three cases, by using a fuzzy order function (σ), "*principle decision-making"* takes place, in the sense that the properties C1 and C2 of the definition of fuzzy order are satisfied.

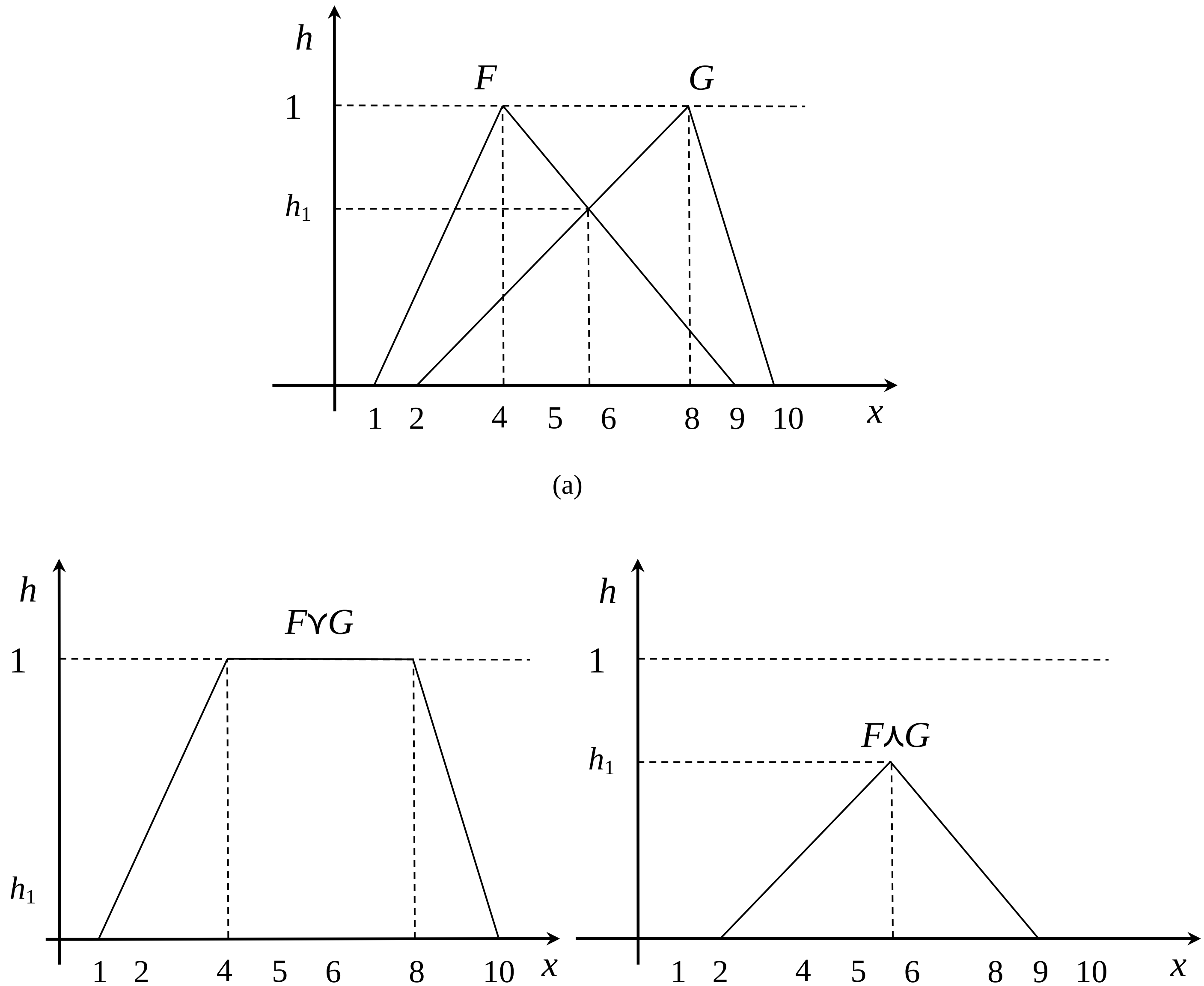


Figure 9.2 *Calculation of join (⋎) and meet (⋏) in lattice (F_1,$\preccurlyeq$) by using membership-function-representations: (a) two IN T1 F and G, (b) join F⋎G and (c) meet F⋏G.*

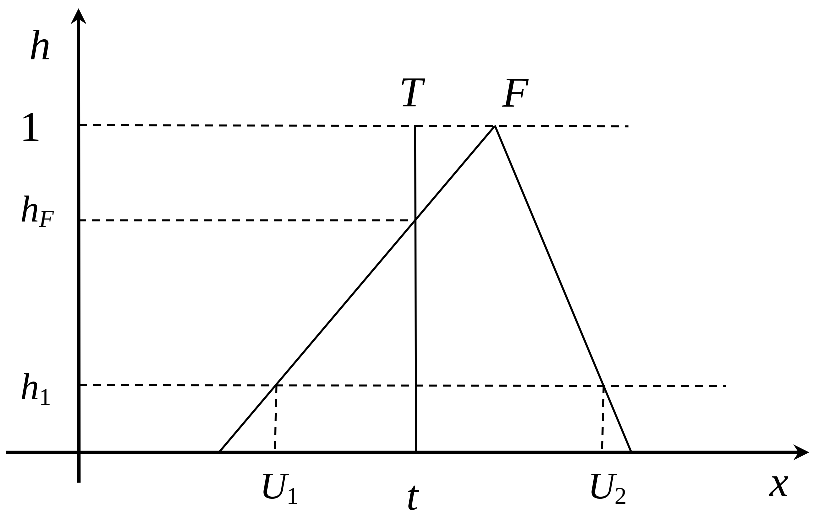


Figure 9.3 *Any IN F∈F$_1$ and a common IN T= [t,t], ∀h∈[0,1] for the explanation of Eq.(9.9) F(x=t) =* $\sigma_\wedge(T,F) = \int_0^1 \sigma_\sqcap(T_h, F_h)\, dh$.

The mathematical interpretation of Eq.(9.9) is considered critical because in combination with fuzzy orders $\sigma_\wedge$(.,.) and σ_Π(.,.) of Section 9.1.6, establishes the use of any function *f*: $R^N \to R$ within the extended CI based on Logic (science), and in particular based on *Fuzzy Lattice Reasoning* (FLR). Note that if the function *f*: $R^N \to R$ is not convex, then it can be approached satisfactory by overlapping convex functions, e.g. Gaussian functions or, more generally, convex functions of any shape.

Assume that the following integral exists, then a metric function D_1: $F_1 \times F_1 \to R_0^+$ between INs T1 is calculated as follows:

$$D_1(F,G) = \int_0^1 d_1(F_h, G_h)\, dh, \tag{9.10}$$

Where function d_1: $I_1 \times I_1 \to R_0^+$ is given by Eq.(9.1).

The *size F* of a IN T1, with hight *hgt*(*F*), is a function Δ_1: $F_1 \to R_0^+$, which is calculated as follows:

$$\Delta_1(F) = \int_0^{hgt(F)} \delta_1(F_h) p(h)\, dh, \tag{9.11}$$

where δ_1: $I_{1p} \to R_0^+$ is a function of *size* of a *common interval T1* and *p*(*h*) is a **probability density** defined in the interval Ω= [0,1], which plays the role of a **weight function**. A special case results for $p(h)= 1$, $h\in[0,1]$. Note that, usually, the hight of an IN T1 is equal to 1, i.e. $hgt(F)= 1$.

9.1.5 Level-4: The Lattice $(\mathsf{F}_2,\preccurlyeq)$ of Intervals' Numbers Type-2 (IN T2)

An **Intervals' Number** (**IN**) **Type-2** (**T2**), or IN T2, is defined as an interval IN T1. That is, an IN T2, by definition, is equal to $[U,W]\doteq \{X\in\mathsf{F}_1\text{: } U \preccurlyeq X \preccurlyeq W\}$, where U is called **lower IN**, and W is called **upper IN** (of IN T2 $[U,W]$).

The set of IN T2 is partially ordered, complete lattice, symbolized as $(\mathsf{F}_2,\preccurlyeq)$. An IN T2 is interpreted as an *information grain* (Kaburlasos & Papadakis 2006).

An IN T2 can, equivalently, represented either by a set of intervals $[U,W]_h$, $h\in[0,1]$, this is the *interval-representation,* or by two functions $U(x)= \bigvee_{h\in[0,1]} \{h\text{: } x \in U_h\}$ and $W(x)= \bigvee_{h\in[0,1]} \{h\text{: } x \in W_h\}$, this is the *membership-function-representations*. Figure 9.4 shows the two different, but equivalent, representations of IN T2.

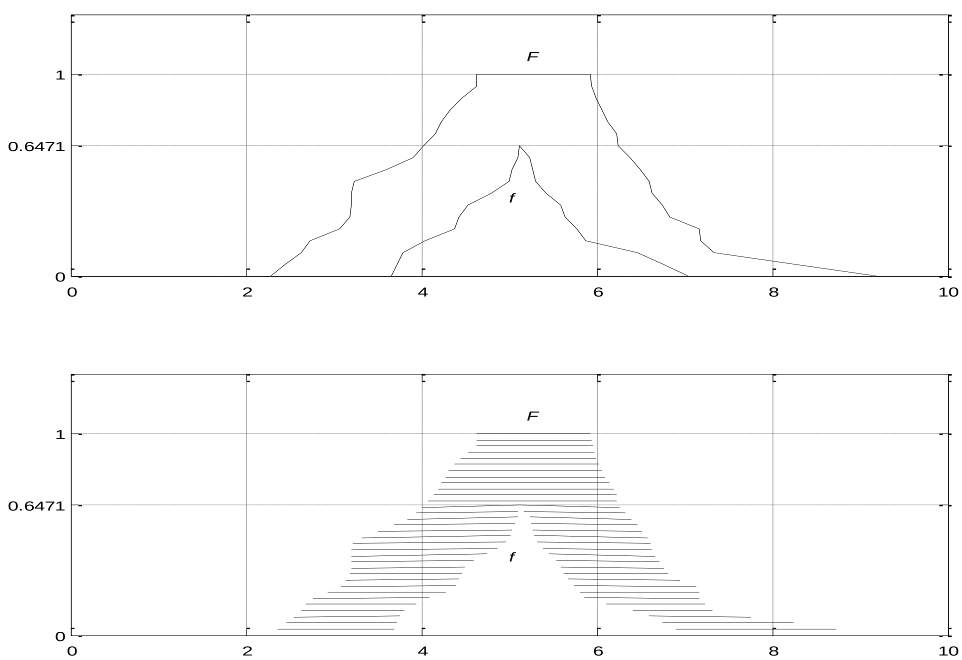


Figure 9.4 *The two equivalent representations of IN T2: (a) the membership-function-representation and (b) the interval-representation.*

The meet ($\curlywedge$) and join ($\curlyvee$) in lattice $(\mathsf{F}_2,\preccurlyeq)$ are given as $(F\curlywedge G)_h = F_h\sqcap G_h$ and $(F\curlyvee G)_h = F_h\sqcup G_h$, respectively, for $h\in[0,1]$. For example, the calculation of join ($\curlyvee$) and meet ($\curlywedge$) in lattice $(\mathsf{F}_2,\preccurlyeq)$ is presented in Figure 9.5 by using membership-function-representations. More specifically, Figure 9.5(α) shows two IN T2 $[f,F]$ and $[g,G]$, where

$f,F,g,G\in$ F$_1$ such that $f\preccurlyeq F$ and $g\preccurlyeq G$. The join $[f,F]\curlyvee[g,G] = [f\curlywedge g,F\curlyvee G]$ is shown in Figure 9.5(b), where $(f\curlywedge g)_h = \varnothing$ for every $h\in(h_1,1]$. Figure 9.5(c) illustrates the meet $[f,F]\curlywedge[g,G] = [f\curlyvee g,F\curlywedge G]$, όπου $(f\curlyvee g)_h = \varnothing$ for every $h\in(h_3,1]$ as well as $(F\curlywedge G)_h = \varnothing$ for every $h\in(h_4,1]$.

Two fuzzy order functions $\sigma_\curlywedge$: F$_2\times$F$_2\rightarrow[0,1]$ and $\sigma_\curlyvee$: F$_2\times$F$_2\rightarrow[0,1]$ can be defined by using Eq.(9.7) and Eq.(9.8) based on fuzzy order functions $\sigma_\sqcap$: I$_2\times$I$_2\rightarrow[0,1]$ and $\sigma_\sqcup$: I$_2\times$I$_2\rightarrow[0,1]$, the last two are given by Eq.(9.5) and Eq.(9.6), respectively.

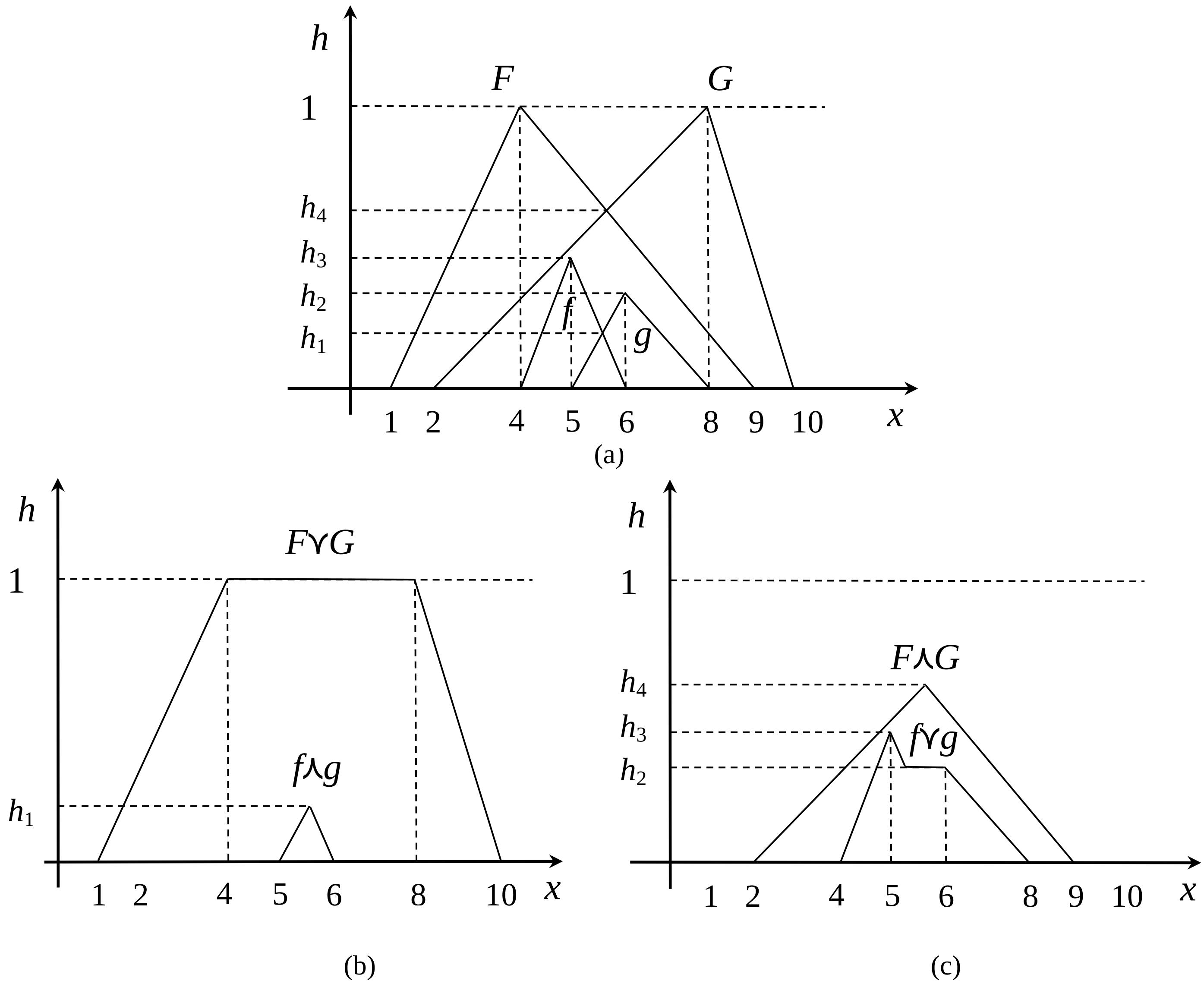


Figure 9.5 *Calculation of join ($\curlyvee$) and meet ($\curlywedge$) in lattice (F_2,$\preccurlyeq$) by using the membership-function-representations (a) two IN T2 [f,F] and [g,G], where $f,F,g,G\in F_1$ such that $f\preccurlyeq F$ and $g\preccurlyeq G$, (b) join $[f,F]\curlyvee[g,G] = [f\curlywedge g,F\curlyvee G]$ and (c) meet $[f,F]\curlywedge[g,G] = [f\curlyvee g,F\curlywedge G]$.*

Assume that the following integal exists, a metric function D_2: $F_2 \times F_2 \rightarrow R_0^+$ between IN T2 is calculated as follows:

$$D_2(F,G)= \int_0^1 d_2(F_h, G_h)dh \qquad (9.12)$$

Where function d_2: $I_2 \times I_2 \rightarrow R_0^+$ is given by Eq.(9.4).

The *size* of IN T2, assume $F= [U,W]$, is a function Δ_2: $F_2 \rightarrow R_0^+$,calculated as follows:

$$\Delta_2(F) = \Delta_1(W) - \Delta_1(U), \qquad (9.13)$$

Where function Δ_1: $F_1 \rightarrow R_0^+$ calculates the size IN T1 by using Eq.(9.11).

9.1.6 Level-5: Lattice of N-groups of Intervals' Numbers T1/T2

Assume the Cartesian product $G= G_1 \times \ldots \times G_N$, ówhere each of the lattices $(G_i, \preccurlyeq)$, $i \in \{1,\ldots,N\}$ is equal to $(F_1, \preccurlyeq)$. Given of two functions ${}^iv$: $L_i \rightarrow R_0^+$ and ${}^i\theta$: $L_i \rightarrow L_i$ to the corresponding lattice $(L_i, \leq)$ of real numbers as describes in Level-0, results a positive valuation function ${}^iv_1([a,b])= {}^iv({}^i\theta(a))+ {}^iv(b)$ in lattice $(L_i \times L_i, \geq \times \leq)$ of generalized intervals and, finally, follow metric functions and fuzzy order functions in lattice $(G, \sqsubseteq)$, as described below. Particularly, fuzzy order function can be defined with two different ways in lattice $(G, \sqsubseteq)$, as explained below.

First, for every $h \in [0,1]$ N-dimensional *cuboids* are assumed in complete lattice $(I_1^N, \sqsubseteq)$. In this way, it results (Kaburlasos & Papadakis 2009):

$$\sigma_{\sqcup}(\boldsymbol{F},\boldsymbol{E}) = \int_0^1 \frac{\sum_{i=1}^N \ {}^iv_1((E_i)_h)}{\sum_{i=1}^N \ {}^iv_1((F_i \curlyvee E_i)_h)} dh, \text{ and} \qquad (9.14)$$

$$\sigma_{\sqcap}(\boldsymbol{F},\boldsymbol{E}) = \int_0^1 \frac{\sum_{i=1}^N \ {}^iv_1((F_i \curlywedge E_i)_h)}{\sum_{i=1}^N \ {}^iv_1((F_i)_h)} dh. \qquad (9.15)$$

Secondly, every dimension in lattice $(F_1^N, \sqsubseteq)$ is assumed separately. Thus, in every dimension $i \in \{1,\ldots,N\}$ it can be defined a fuzzy order σ_i: $F_1 \times F_1 \rightarrow [0,1]$ according to Eq.(9.7) or Eq.(9.8). Finally, a fuzzy order σ_c: $F^N \times F^N \rightarrow [0,1]$ in lattice $(F^N, \sqsubseteq)$ can be defined with the *convex combination* $\sigma_c(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N))= \lambda_1\sigma_1(F_1,E_1)+\ldots+\lambda_N\sigma_N(F_N,E_N)$, where $\lambda_1,\ldots,\lambda_N \geq 0$ such that $\lambda_1+\ldots+\lambda_N = 1$. Two other fuzzy orders are calculated from (a) $\sigma_{\wedge}(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N))= \min_{i \in \{1,\ldots,N\}} \sigma_i(F_i, E_i)$ and (b) $\sigma_{\Pi}(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N))= \prod_{i=1}^N \sigma_i(F_i, E_i)$, respectively (Kaburlasos & Kehagias, 2014). In other words, fuzzy order $\sigma_{\wedge}(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N))$ is equal to the minimum of fuzzy order $\sigma_i(F_i,E_i)$, $i\in\{1,...,N\}$, while fuzzy order $\sigma_\Pi(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N))$ is equal to the product of fuzzy orders $\sigma_i(F_i,E_i)$, $i\in\{1,...,N\}$.

Moreover, *metric* functions D: G$\rightarrow R_0^+$ in lattice (G,⊑) are calculated as follows:

$$D(\boldsymbol{F}= (F_1,...,F_N), \boldsymbol{E}= (E_1,...,E_N)) = [(D_1(F_1,E_1))^p + \cdots + (D_1(F_N,E_N))^p]^{1/p}, \quad (9.16)$$

where p∈R, while metric D_1: $F_1\times F_1\rightarrow R_0^+$ υis calculated from Eq.(9.10).

The *size* of a N-dimensional IN T1 $A= (A_1,\ldots,A_N)$ is calculated from the *convex combination*:

$$\Delta(A) = p_1\Delta_1(A_1)+\ldots+p_N\Delta_1(A_N) \quad (9.17)$$

as a specific application of the general definition of size function in a poset (see in Chapter 7) with Ω= {1,…,N}. Moreover, note that function $\Delta_1: F_1 \rightarrow R_0^+$ in Eq.(9.17) is calculated according to Eq.(9.11), as another specific application of the aforementioned general definition of the *size* function, with Ω= [0,1].

All equations in this Section are easily extended to N-dimensional IN T2.

9.1.7 Further Extensions

In the above-mentioned hierarchy, at least one additional level can be introduced, and thus generalizing an IN T1/T2 as described below. In particular, an IN T1/T2 has a two-dimensional (2-D) representation at the level, which can be generalized in three dimensions (3-D) as explained below (Kaburlasos & Papakostas, 2015).

A 3-D IN T1 (respectively, T2) is defined as a function F: [0,1]→F, where F= F_1 (respectively, F= F_2), satisfies the relation $z_1 \leq z_2 \Rightarrow F_{z_1} \succcurlyeq F_{z_2}$. In other words, a 3-D IN T1 (respectively, T2) F has a three-dimensional representation F_z such that for a constant $z= z_0$ functionF_{z_0}, which is called **zSlice**, is a 2-D IN T1 (respectively, T2). For example, Figure 9.6(α) illustrated a 3-D IN T2 F_z, $z\in[0,1]$, which is intersected by the plane $z= 0.5$. The zSlice $F_{0.5}$ is a 2-D IN T2, which is shown in Figure 9.6(β).

Assume that F_g symbolize either the set of 3-D IN T1 or the set of 3-D IN T2. In eny case, the duet (F_g,≼) is a lattice with order $E \sqsubseteq F \Leftrightarrow E_z \preccurlyeq F_z$, for every $z\in[0,1]$. A function σ_{F_g}: $F_g\times F_g\rightarrow[0,1]$ of fuzzy order in lattice (F_g,≼) is defined as:

$$\sigma_{F_g}(E,F) = \int_0^1 \int_0^1 \sigma_I\,((E_z)_h, (F_z)_h) dh dz, \quad (9.18)$$

where function $\sigma_I(.,.)$ is given from one of the equations (9.2), (9.3), (9.5), (9.6).

9.2 Intervals' Numbers (INs)

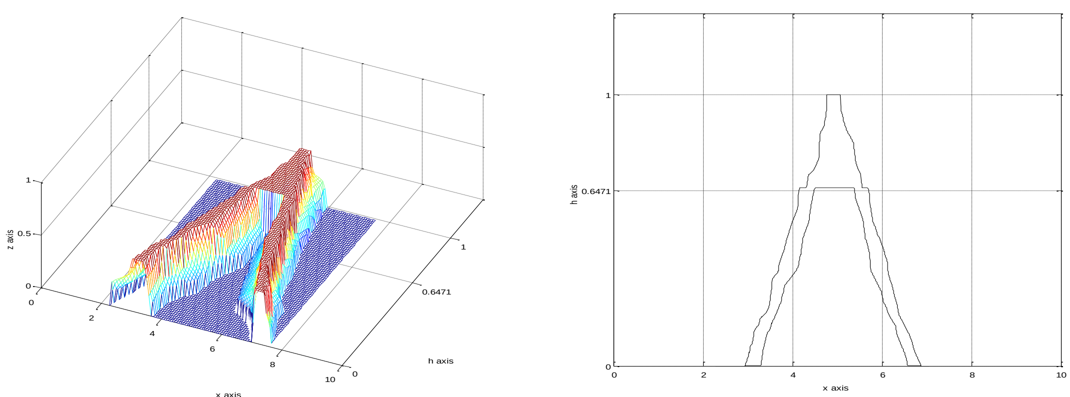
This section focuses on Intervals' Numbers Type-1, or INs (T1) in brief.

9.2.1 Interpretations IN

Several associations have been proposed in the literature between **possibility (distribution**) and **probability (distribution)** (Ralescu & Ralescu, 1984; Wonneberger, 1994). In this book an IN is a "mathematical object" which can be interpreted in, at least, two different ways. In particular, firstly, an IN can be interpreted as a fuzzy number, which represents a *possibility distribution*, and second, an IN can be interpreted as a *probability distribution* as explained below.

Assume a probability distribution according to which we choose the population of samples (see real numbers) shown in Figure 9.7(a). Let M = 5.96 the value of the **median** of the sample, that is, the value that divides the sum of the sample values into two equal parts. Figure 9.7(b) shows the corresponding, strictly increasing **cumulative distribution function** *c*(.) with c (5.96) = 0.5. Figure 9.7(c) shows the calculation of an IN *E* in the following way: For every $x \leq$ M= 5.96, is computed the function 2*c*(*x*), while for each $x >$ M= 5.96 is computed the function 2(1-*c*(*x*)).

The aforementioned algorithm is called "CALCIN" and calculates the IN *E* shown in Figure 9.7(c) with membership-function-representation *E(x)*, which is interpreted as a probability distribution. In particular, if *F* is an IN, calculated with the CALCIN algorithm, then the interval *F(h)* includes 100(1-*h*)% of the distribution, while the remaining 100*h*% is equally distributed below- and above- *F(h)* (Kaburlasos, 2004; Kaburlasos, 2006; Kaburlasos & Pachidis, 2014; Kaburlasos & Papadakis, 2006; Papadakis & Kaburlasos, 2010).



(a) (b)

Figure 9.6 *(a) A 3-D Numbers' Interval Type-2 (3-D IN T2), assume F.*
(b) The zSlice $F_{0.5}$ (of 3-D IN T2 F) is the 2-D IN T2 of the figure.

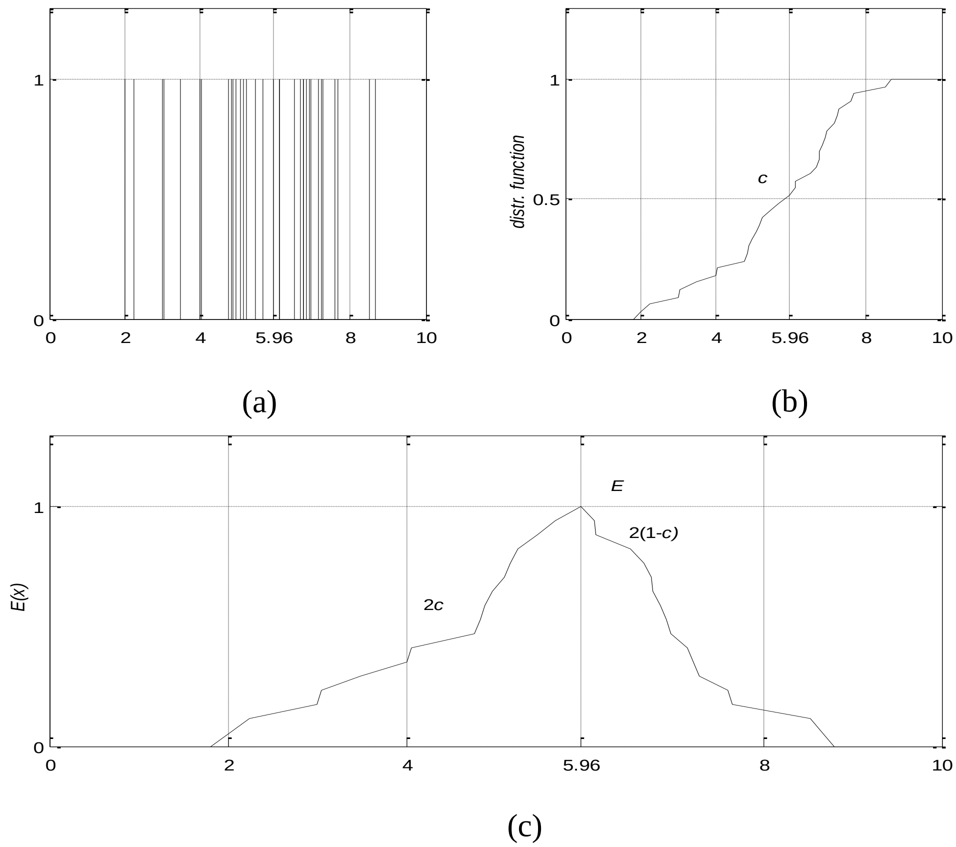


Figure 9.7 *(a) A distribution of samples, see real numbers, in the interval [0, 10], with median value M= 5.96,*
(b) the corresponding cumulative distribution function c(.),
(c) calculation of IN E from distribution function c(.) according to algorithm CALCIN.

Of particular interest is an IN of the form [*a*,*b*], for every $h\in[0,1]$, which represents the interval *[a,b]* of real numbers. Thus, in N dimensions, is formed the space of **cuboids** (Kaburlasos, 2006), which (space) has drawn the research interest (Dietterich et al., 1997; Long & Tan 1998; Salzberg, 1991; Samet, 1988) thanks to its simplicity and efficiency in computational applications. Note also that the *fuzzy order* function was originally presented under the name **inclusion measure** because it was only used with cuboids instead of the more general N-dimensional IN. Later it was expanded to grids, so the original name (see inclusion measure) have changed to *fuzzy order*.

9.2.2 Representation IN

From a practical point of view, an IN *F* is represented in the computer memory as an L×2 table $[a_1\ b_1;\ a_2\ b_2;\ldots;\ a_L\ b_L]$ of real numbers, where L is the pre-defined by the user number of levels h_1, h_2,…, h_L so that $0<h_1\leq h_2\leq\ldots\leq h_L=1$. In practice, it is usually used L=16, or L=32, levels per equal intervals, in the interval [0,1]. Note that a number of 16, or 32 layers has also been proposed in applications of Fuzzy Inference Systems (FIS) based on fuzzy numbers α-cuts (Kaburlasos & Kehagias, 2014; Uehara & Fujise, 1993; Uehara & Hirota, 1998).

As a consequence of the above assumptions, a 2-D IN T2, assume [U, W], is represented by a L×4 table because for each of the "L" levels along the *h* axis, two intervals are stored: an interval for the lower *IN U*, and an interval for the upper *IN W*. Finally, a 3-D IN T2 is represented by a L×4×L table because for each of the "L" levels along the z-axis a 2-D IN T2 is stored.

9.2.3 Calculations with INs

Several *fuzzy numbers arithmetics* have been proposed in the literature, some of which are based on **intervals' arithmetic** (Kaufmann & Gupta, 1985; Moore & Lodwick, 2003). This section introduces a new arithmetic in INs, based on a new intervals' arithmetic with fewer algebraic constraints for greater computational flexibility, as explained below.

Assume the complete lattice (L=[-∞,+∞],≤) of real numbers. The lattice (L×L,≥×≤) of *generalized intervals* is a **linear space** (Papadakis & Kaburlasos, 2010) because addition and multiplication can be defined, as explained below:

The *addition* between generalized intervals is defined as follows:

$$[a,b] + [c,e] = [a+c, b+e],$$

while the *multiplication* of a generalized interval and a real number is defined as follows:

$$k[a,b] = [ka, kb].$$

Addition and multiplication can be extended in the set G of GIN. Specifically, if $F,H\in$G then addition is defined as $F+H = F_h+H_h$, where $h\in[0,1]$, and multiplication of a GIN with a real number k is defined as $kF = kF_h$, where $h\in[0,1]$. It results that (G,⊑) of GIN Eis a linear space.

It is interesting that $[a,b],[c,e]\in I_1$ implies that $[a+c, b+e]\in I_1$. However, δgiven of an interval T1 $[a,b]\in I_1$ and a real number $k\in$R, there is no guarantee that the product $k[a,b]$ will be an interval T1. In particular, for a negative number $k<0$, the product $k[a,b]$ is not an interval T1 when $ka > kb$.

In the above-mentioned context, it has been shown that the set of T1 intervals is a **cone** in the linear space of generalized intervals. Recalled that, by definition, a cone is called a subset C of linear space if and only if for $x_1,x_2\in C$ and for non-negative numbers

$\lambda_1,\lambda_2 \geq 0$ the linear combination $(\lambda_1 x_1+\lambda_2 x_2)$ belongs to *C*. Based on the above results that the set of intervals T1 is a cone in the linear space of generalized intervals T1.

Extensions can also be made in the interval F_1 of the IN. In particular, the addition of two INs *F* and *G* can be defined as $(F + G)=F_h + G_h$,, where $h\in[0,1]$, while the multiplication of a IN *F* with a real number *k* can be defined as $kF = kF_h$, where $h\in[0,1]$. Similarly, as before, it follows that interval F_1 of the IN is a cone in the linear space of the GIN (Papadakis & Kaburlasos, 2010). It should be noted that the partial order relationship of IN has already been studied on the basis of membership-function-representation (Zhang & Hirota, 1997) where fuzzy numerical operations are defined between INs. This book proposes analysis of INs based on interval-representation, while numerical operations are defined within a linear space, followed by well-known algebraic practices (Luxemburg & Zaanen, 1971; Vulikh, 1967).

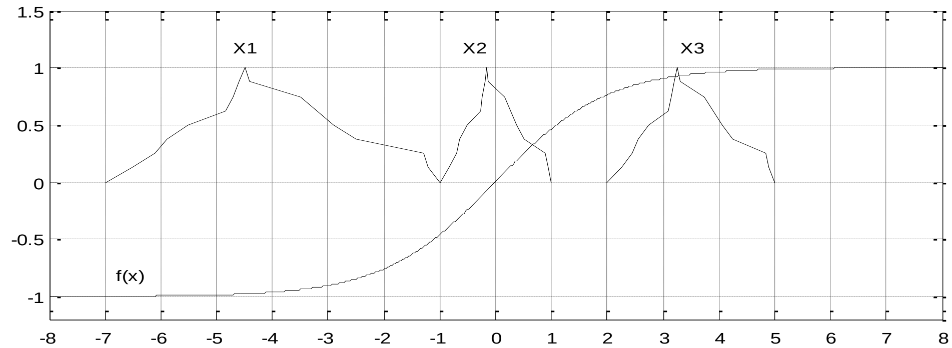
We can introduce a non-linear transformation (Kaburlasos et al., 2013) into the space of intervals T1 by considering a *strictly increasing real function f:* R→R. In particular, a space T1 $[a,b]$ is transformed into the interval T1 $[f(a), f(b)]$. Extensions can also be made in the interval F_1 of IN, where given of a strictly increasing real function f: R→R, one IN is transformed into another IN as follows $f(F)]_h = f(F_h)$, $h\in[0,1]$. For example, Figure 9.8(a) depicts a strictly increasing function, this is the sigmoid function $f(x) = (1-e^{-x})/(1+e^{-x})$ which transforms the IN X1, X2 and X3 of Figure 9.8(a) into Y1 = f (X1), Y2 = f (X2) and Y3 = f (X3) of Figure 9.8 (b), respectively.

When we choose, in a particular application, to interpret each IN as a probability distribution, then all the aforementioned (non-linear) transformations and operations between IN can be performed between population of measurements through regression algorithms as explained below.

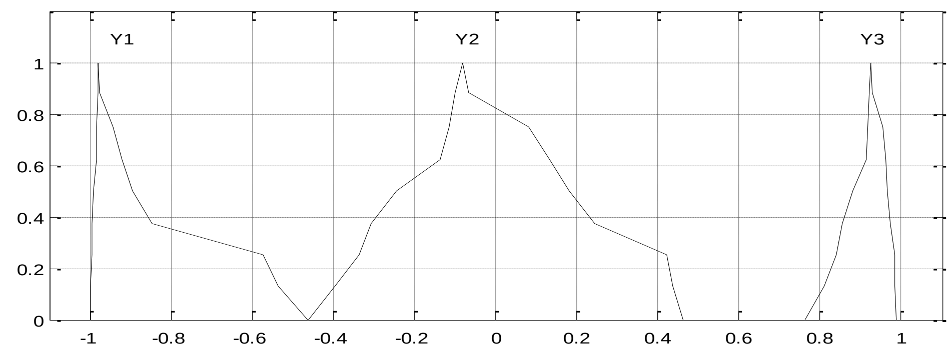
The following computational example interprets geometrically, the *consistency property* (C2) (see the definition of fuzzy order) in the Cartesian product $([0,1],\leq)\times([0,1],\leq)$ as shown in Figure 9.9(a) and (b), with the same functions $v(x)= x$ and $\theta(x)= 1-x$ per basic matrix $([0,1],\leq)$. In addition, Figure 9.9(a) and (b) shows two "boxes" $u= [0.5,0.6]\times[0.3,0.4]$ and $w= [0.4,0.9]\times[0.2,0.8]$ with $u\sqsubseteq w$. Note that Figure 9.9 (a), besides *u* and *w*, shows the "box" $x= [0.15,0.2]\times[0.15,0.2]$, which is outside *u* and *w*. Whereas, in addition to *u* and *w*, Figure 9.9(b) shows the "box" $x'= [0.85,0.9]\times[0.55,0.6]$, which lies outside *u*, but within *w*.

It is easy to verify the calculations $x\sqcup u= [0.15,0.6]\times[0.15,0.4]$ and $x\sqcup w= [0.15,0.9]\times[0.15,0.8]$, as well as the calculations $x'\sqcup u= [0.5,0.9]\times[0.3,0.6]$ and $x'\sqcup w= w$. Subsequently, we calculate fuzzy order $\sigma_\sqcup(.,.)$. In particular, it results that $\sigma_\sqcup(x,u)= \frac{V(u)}{V(x\sqcup u)}= \frac{v(\theta(0.5))+v(0.6)+v(\theta(0.3))+v(0.4)}{v(\theta(0.15))+v(0.6)+v(\theta(0.15))+v(0.4)}= \frac{2.2}{2.7}\approx 0.8148$ and $\sigma_\sqcup(x,w)= \frac{3.1}{3.4}\approx 0.9118$. Thus, the inequality $\sigma_\sqcup(x,u)\leq\sigma_\sqcup(x,w)$ is validated in Figure 9.9(a). Additionally, it results that

$\sigma_{\sqcup}(x',u)= \frac{V(u)}{V(x'\sqcup u)}= \frac{v(\theta(0.5))+v(0.6)+v(\theta(0.3))+v(0.4)}{v(\theta(0.5))+v(0.9)+v(\theta(0.3))+v(0.6)}= \frac{2.2}{2.7}\approx 0.8148$ and $\sigma_{\sqcup}(x',w)= 1$. Thus, the inequality $\sigma_{\sqcup}(x',u)\leq\sigma_{\sqcup}(x',w)$ is validated in Figure 9.9(b).



(a)



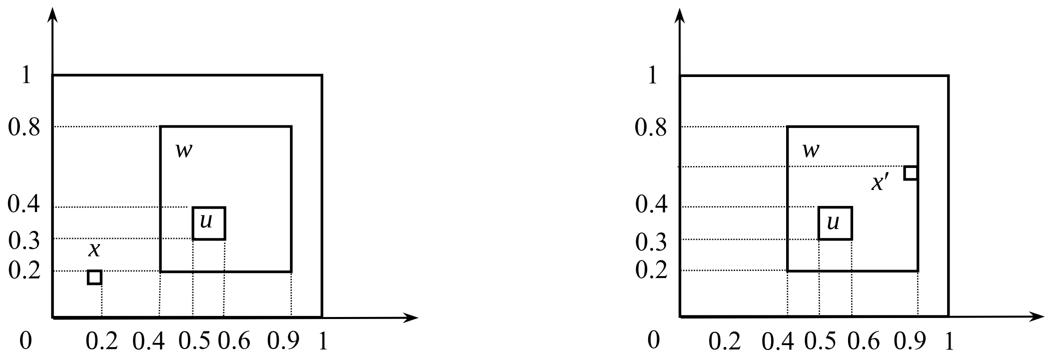
(b)

Figure 9.8 *(a) The sigmoid function $f(x) = (1-e^{-x})/(1+e^{-x})$ and three INs X1, X2 and X3. (b) The domain [0,1] of images Y1= f(X1), Y2= f(X2) and Y3= f(X3) by definition is equal to the domain [0,1] of sigmoid $f(x) = (1-e^{-x})/(1+e^{-x})$.*

As a follow-up to the calculations, observe that in some applications "missing data" and/or "don't care data" may appear in a *basic lattice*, where by saying "missing data" means the absence of a particular value in a basic lattice, and by saying "don't care data" we mean the presence of all possible values in a basic lattice. For a substantial way of manipulating the above data, we represent (a) a "missing data" with the least element $[i,o]$, and (b) a "don't care data" with the greatest element in the corresponding basic lattice $(L=[o,i],\leq)$. Next, the "missing data" and/or "don't care data" are presented to verify the *property of consistency* (C2).

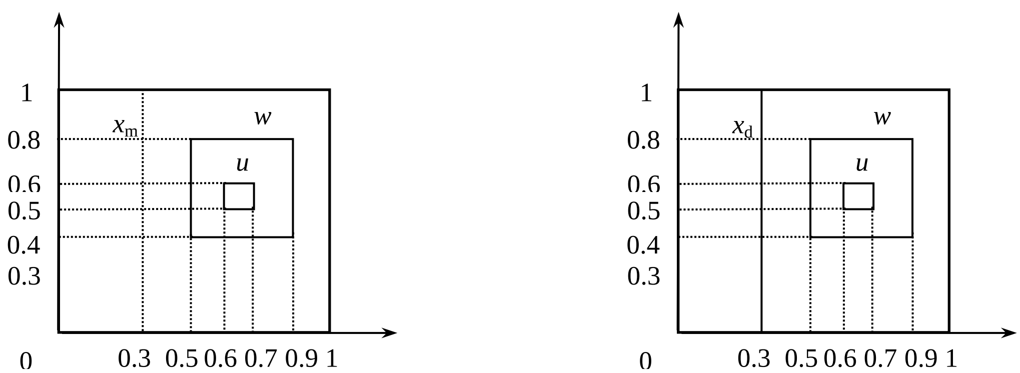
Assume the Cartesian product $([0,1],\leq)\times([0,1],\leq)$ of two basic lattices $([0,1],\leq)$ with same functions $v(x)= x$ and $\theta(x)= 1-x$ per basic lattice, as shown in Figure 9.10(a) and (b). Moreover, Figure 9.10(a) and (b) depicts two "boxes" $u= [0.6,0.7]\times[0.5,0.6]$ and

w= [0.5,0.9]×[0.4,0.8] with $u \sqsubseteq w$. Figure 9.10(a) shows the data x_m= [0.3,0.3]×[1,0] with “missing data” in the second basic lattice, while Figure 9.10(b) shows the data x_d= [0.3,0.3]×[0,1] with “don’t care data” in the second basic lattice.



(a) (b)

Figure 9.9 *(a) and (b): The property of Consequence «$u \sqsubseteq w \Rightarrow \sigma(x,u) \leq \sigma(x,w)$» guarantees that when a box u is inside of another w then every box x (ή x′) is contained, in the sense of a fuzzy order, more in w rather than in u.*



(a) (b)

Figure 9.10 *The property of Consequence «$u \sqsubseteq w \Rightarrow \sigma(x,u) \leq \sigma(x,w)$» is valid with "missing data" and/or "don’t care data".*
(a) "missing data" along the vertical axis, that is x_m= [0.3,0.3]×[1,0].
(b) "don’t care data" along the vertical axis, that is x_d= [0.3,0.3]×[0,1].

Validation of the calculations $x_m \sqcup u$= [0.3,0.7]×[0.5,0.6], $x_m \sqcup w$= [0.3,0.9]×[0.4,0.8], $x_d \sqcup u$= [0.3,0.7]×[0,1] and $x_d \sqcup w$= [0.3,0.9]×[0,1] is easy. Στη συνέχεια, υπολογίζουμε βαθμούς διάταξης $\sigma_{\sqcup}(.,.)$. In particular, it results $\sigma_{\sqcup}(x_m,u)$= $\frac{V(u)}{V(x_m \sqcup u)} = \frac{v(\theta(0.6))+v(0.7)+v(\theta(0.5))+v(0.6)}{v(\theta(0.3))+v(0.7)+v(\theta(0.5))+v(0.6)} = \frac{2.2}{2.5} \approx 0.8800$ and $\sigma_{\sqcup}(x_m,w)= \frac{2.8}{3.0} \approx 0.9333$. Thus, the inequality $\sigma_{\sqcup}(x_m,u) \leq \sigma_{\sqcup}(x_m,w)$ is validated in Figure 9.10(a). Moreover, it results

$\sigma_{\sqcup}(x_d,u)= \frac{V(u)}{V(x_d \sqcup u)} = \frac{v(\theta(0.6))+v(0.7)+v(\theta(0.5))+v(0.6)}{v(\theta(0.3))+v(0.7)+v(\theta(0))+v(1)} = \frac{2.2}{3.4} \approx 0.6471$ and $\sigma_{\sqcup}(x_d,w)= \frac{2.8}{3.6} \approx 0.7778$. Thus, the inequality $\sigma_{\sqcup}(x_d,u) \leq \sigma_{\sqcup}(x_d,w)$ is validated in Figure 9.10(b).

9.3 Implementation Algorithms IN

Algorithms that calculate with IN have already been suggested in the bibliography either for machine learning or for regression as explained below.

9.3.1 Machine Learning Algorithms

Remember that every use of a fuzzy order function (σ) is called Fuzzy Lattice Reasoning (FLR) (Kaburlasos & Kehagias, 2014). Three algorithms are presented in pseudo-code, where the first algorithm is for clustering (Figure 9.11), the second algorithm is for classification (Figure 9.12), while the third algorithm is for identification (Figure 9.13), all in the IN interval. Note that the aforementioned algorithms are generalizations of corresponding algorithms of the *Adaptive Coordination Theory* from the N-dimensional Euclidean R^N space to the F^N.

Pattern recognition (Duda et al., 2001) is an area of interest in the literature. Observe that the above algorithms, and/or variations of the above-mentioned algorithms, have already been extensively used in pattern recognition problems (Kaburlasos, 2004; Kaburlasos & Papadakis, 2009; Kaburlasos et al., 2012). In fact, note that some variations are based on a distance metric function instead of relying on a fuzzy order function. A selection criterion for functions $v(.)$ and $\theta(.)$ in Level-0 are presented next.

One of the older applications of FLR algorithm for classification, where the N-dimensional INs $W_1,\ldots,W_{|C_a|}$ were exclusively *cuboids* and the input $\boldsymbol{X}_i$ was exclusively a point (i.e. trivial cuboid) in the N-dimensional Euclidean space, lead to the selection of a pair of functions $v(.)$ and $\theta(.)$ such that the condition $v_1([a,a])= v(\theta(a))+v(a)= 1$ to be valid in every basic lattice based on the following reasoning. As a result of the aforementioned condition in every basic lattice it results that $v_1([a,b])= v(\theta(a))+v(b)= v(b)-[1-v(\theta(a))]+1= [v(b)-v(a)]+1= \delta_1([a,b])+1$. By using the positive valuation function $V([a_1,b_1]\times\ldots\times[a_N,b_N])= v_1([a_1,b_1])+\ldots+ v_1([a_N,b_N])$ for every cuboid, it results that: $\sigma_{\sqcup}(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i)= \frac{V(X_i)}{V(W_J \sqcup X_i)} = \frac{N}{N+\Delta(W_J \sqcup X_i)}$.

Line 16 in Figure 9.12 assimilates the input $\boldsymbol{X}_i$ recalculating $\boldsymbol{W}_J$ according to the relation $\boldsymbol{W}_J \doteq \boldsymbol{W}_J \sqcup \boldsymbol{X}_i$ when already condition $\sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) \geq \bar{\rho}_a$ is fulfilled. The latter (condition) applies $\frac{N}{N+\Delta(W_J \sqcup X_i)} \geq \bar{\rho}_a \Leftrightarrow \Delta(W_J \sqcup X_i) \leq \frac{N(1-\bar{\rho}_a)}{\bar{\rho}_a}$. In other words, the calculation $\boldsymbol{W}_J \doteq \boldsymbol{W}_J \sqcup \boldsymbol{X}_i$ is realized only when the *size* $\Delta(\boldsymbol{W}_J \sqcup \boldsymbol{X}_i)$ of the cuboid $\boldsymbol{W}_J \sqcup \boldsymbol{X}_i$ is up to $\frac{N(1-\bar{\rho}_a)}{\bar{\rho}_a}$. For all the reasons above, FLR algorithm for classification, even when applied to the generalized lattice $(F_1^N,\sqsubseteq)$ or to lattice $(F_2^N,\sqsubseteq)$, usually checks the condition

"$\Delta(\boldsymbol{W}_J \sqcup \boldsymbol{X}_i) \leq \bar{\Delta}_a$" instead of the condition "$\sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) \geq \bar{\rho}_a$" so as to further calculate $\boldsymbol{W}_J \doteq \boldsymbol{W}_J \sqcup \boldsymbol{X}_i$.

A pair of functions $v(.)$ and $\theta(.)$, which satisfies the condition $v_1([a,a])= v(\theta(a))+v(a)= 1$ in the basic lattice (L=[0,1],≤) is the $v(x)= x$ and $\theta(x) = 1-x$. Another pair, which satisfies the condition $v_1([a,a])= v(\theta(a))+v(a)= 1$ in lattice (L=[-∞,+∞],≤) is the $v(x)= \frac{1}{1+e^{-\lambda(x-\mu)}}$ and $\theta(x) = 2\mu-x$. Note that function $v(x)= \frac{1}{1+e^{-\lambda(x-\mu)}}$ is known (α) in statistical analysis under the name **logistics** (Kleinbaum & Klein, 2002) and (β) in ANN with the name **sigmoid** (Duda et al., 2001). The mathematical analysis here, have presented an advantage of the aforementioned $v(.)$ function in the domain of IN.

Assume a set $\{W_1, \ldots, W_{|C|}\} = C \subset 2^{F_1^N}$, assume $K = |C|$ the *cardinality* of set C, and assume the **vigilance parameter** $\rho \in [0,1]$ which is defined by the user.

From $i = 1$ to $i = n_{trn}$ do

  consider the next input (AΔ) $\boldsymbol{X}_i \in F_1^N$.

  Assume the set $S \doteq C$.

  $J = \underset{\substack{j \in \{1,\ldots,|S|\} \\ W_j \in S}}{argmax}[\sigma(X_i \sqsubseteq W_j)]$.

  For as it applies $(S \neq \{\})$.AND.$(\sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) < \rho)$ do

    $S \doteq S \setminus \{\boldsymbol{W}_J\}$.

    $J = \underset{\substack{j \in \{1,\ldots,|S|\} \\ W_j \in S}}{argmax}[\sigma(X_i \sqsubseteq W_j)]$

  End // For as it applies …

  If $S \neq \{\}$ then

    $C \doteq C \cup \{\boldsymbol{X}_i\}$.

    $K \doteq K+1$.

  else

    $\boldsymbol{W}_J \doteq \boldsymbol{W}_J \sqcup \boldsymbol{X}_i$.

  End // If …

End // From … to …

Figure 9.11 *Fuzzy Lattice Reasoning (FLR) Algorithm for clustering.*

Assume the set $\{W_1,\ldots,W_{|C_a|}\} = C_a \subset 2^{F_1^N}$, assume $K = |C_a|$ the *cardinality* of set C_a, assume the *vigilance parameter* $\bar{\rho}_a \in [0,1]$ which is defined by the user, assume ε a small positive number, assume $B = \{b_1,\ldots,b_L\}$ a set of "labels", and assume the function $\ell: F_1^N \to B$ on C_a.

From $i = 1$ to $i = n_{trn}$ do

  Assume the next input $(\boldsymbol{X}_i, \ell(\boldsymbol{X}_i)) \in F_1^N \times B$.

  Assume the set $S \doteq C_a$.

  $J = \underset{\substack{j\in\{1,\ldots,|S|\} \\ W_j \in S}}{argmax}[\sigma(X_i \sqsubseteq W_j)]$.

  If $\ell(\boldsymbol{W}_J) \neq \ell(\boldsymbol{X}_i)$ then $\bar{\rho}_a = \sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) + \varepsilon$.

  For as it applies $(S \neq \{\})$.AND.$(\sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) < \bar{\rho}_a)$ do

    $S \doteq S \backslash \{\boldsymbol{W}_J\}$.

    $J = \underset{\substack{j\in\{1,\ldots,|S|\} \\ W_j \in S}}{argmax}[\sigma(X_i \sqsubseteq W_j)]$

    If $\ell(\boldsymbol{W}_J) \neq \ell(\boldsymbol{X}_i)$ then $\bar{\rho}_a = \sigma(\boldsymbol{W}_J \sqsubseteq \boldsymbol{X}_i) + \varepsilon$.

  End // For as it applies …

  If $S \neq \{\}$ then

    $C_a \doteq C_a \cup \{\boldsymbol{X}_i\}$ and $K \doteq K+1$.

    If $\ell(\boldsymbol{X}_i) \notin B$ then $(B \doteq B \cup \{\ell(\boldsymbol{X}_i)\}$ and $L \doteq L+1)$.

  else

    $\boldsymbol{W}_J \doteq \boldsymbol{W}_J \sqcup \boldsymbol{X}_i$.

  End // If …

End // From … to …

Figure 9.12 *Fuzzy Lattice Reasoning (FLR) Algorithm for classification.*

1: Assume a set $\{W_1,\ldots,W_K\} = C \subset 2^{F_1^N}$ of INs, assume $|C|$ the *cardinality* of set C, assume a set $B = \{b_1,\ldots,b_L\}$ from labels, and assume a depiction $\ell\colon F_1^N \to B$.
2: From $i = 1$ to $i = n_{tst}$ do
3: Assume the next input pair $(\boldsymbol{X}_i, b_i) \in F_1^N \times B$ for recognition.
4: $J = \underset{\substack{j \in \{1,\ldots,|C|\} \\ W_j \in C}}{argmax}[\sigma(X_i \sqsubseteq W_j)]$.
5: The input data $\boldsymbol{X}_i$ is classified to class $\ell(\boldsymbol{W}_J)$.
6: End // From … to …
7: Calculate the percentage of correct recognitions in the data set.

Figure 9.13 *Fuzzy Lattice Reasoning (FLR) Algorithm for clustering for recognition.*

9.3.2 Regression Algorithms

Two INs can be added to each other. Also, an IN can be multiplied with a non-negative number and/or transformed non-linearly using a (strictly) increasing function. By performing the two operations mentioned above (addition and multiplication), a regression algorithm arises which shows, non-linearly, an N-group of IN in a IN. For example, consider the ANN of Figure 1.2 with weights that are only positive numbers and IN inputs. According to the above, the ANN of Figure 1.2 can implement a regression function f: $F_1^N \to F_1$ (Kaburlasos, 2013). Therefore, the architecture of Figure 1.2 can implement a neuro-fuzzy system if an IN is interpreted as a *feasibility distribution*. Extensions to IN T2 (Mendel, 2013) are directly feasible.

Further possibilities arise if an IN is interpreted as a *probability distribution*. In the latter case, the architecture of Figure 1.2 can calculate a distribution (at the output of the ANN) of N distributions (at the input of the ANN). Therefore, if an IN represents a (huge) distribution of arithmetic samples then IN manipulation involves the manipulation of **big data** (Kaburlasos & Papakostas, 2015).

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