Convergence results on the stable states of a Gravitational Swarm solving the Graph Coloring Problem

Manuel Graña, Israel Rebollo

Computational Intelligence Group, Computer Science and Artificial Intelligence, Universidad del Pais Vasco.

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Summary

Gravitational Swarm (GS) is a collection of agents endowed with mass moving in the space under the forces generated by their gravitational attraction.

Stable global states of this system

solutions of the Graph Coloring Problem (GCP) into a GS,

formal proofs ensuring the convergence GS to solutions of the GCP on

undirected graphs with a

given number of colors.

computational experiments showing

competitive performance of the approach compared with state-of-the-art GCP solving algorithms.

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Introduction

Graph Coloring Problem (GCP) assigning a color to each vertex of a graph G with the restriction that any pair of vertices linked by an edge can't have the same color.

Chromatic number K is the minimum number of colors needed to color the graph.

Minimal graph coloring: finding the graph's chromatic number, is an NP-complete problem.

Exact algorithms for the solution of the problem are feasible only in very specific cases.

Introduction

- Suboptimal deterministic algorithms,
  - Brelaz greedy algorithm
- Approximate approaches
  - Heuristic local search algorithms: Tabu search
  - Random global search algorithm, Simulated annealing ,
  - Bio-inspired algorithms
    - Ant Colony Optimization (ACO) ,
    - Particle Swarm Optimization (PSO).

Introduction

- Swarm Intelligence approach.
  - assuming that the agents are embedded in a global attraction field
- Gravitational Swarm (GS) where particles are endowed with
  - a mass-like property that is affected by global attractive forces.
  - electrical charge selectively affected by local repulsive forces.

Introduction

The mapping of the GCP into a GS consists in the following correspondences:

- Graph vertices correspond to agents in the GS.
- Graph edges correspond to repulsive relations between agents.
- Colors are modelled as special static agents attracting all moving vertex agents.
- The vertex agents reaching a color agent position acquire its color, and stay at this position unless repelled by another vertex agent.
- When all the vertex agents are in some color agent position, without any repulsive force exerted between any pair of vertices, we say that the GS has reached a stable state.
- Stable states can be readed as GCP solutions.

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Graph Coloring

Definitions

An undirected graph is a collection of vertices linked by edges $G = (V, E)$, such that $V = \{v_1, \ldots, v_N\}$ and $E \subseteq V \times V$, and $(v, w) \in E \Rightarrow (w, v) \in E$.
The neighborhood of a vertex in the graph is the set of vertices linked to it: $\mathcal{N}(v) = \{w \in V \,|(v, w) \in E\}$.

Definition

Graph coloring. Let $C = \{c_1, \ldots, c_M\}$ denote a set of colors. Given a graph $G = (V, E)$, a graph coloring is a mapping of graph vertices to colors $\mathcal{C} : V \to C$ such that no two neighboring vertices have the same color, i.e. $w \in \mathcal{N}(v) \Rightarrow \mathcal{C}(v) \neq \mathcal{C}(w)$.

Graph Coloring

Definition

Minimal graph coloring. A set of colors C^* is minimal relative to graph $G = (V, E)$ if (1) there is a graph coloring $\mathcal{C}^* : V \to C^*$, and (2) for any smaller set of colors there is no graph coloring using it:

$$|C| < |C^*| \Rightarrow \neg\exists\mathcal{C} : V \to C$$

.

Definition

Chromatic number: The chromatic number M^* is the number of colors of the minimal graph coloring C^*.

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Gravitational Swarm

Definition

A Gravitational Swarm (GS) is a collection of particles $P = \{p_1, \ldots, p_L\}$ moving in an space $\mathcal{S}$ subjected to atractive and repulsive forces. Particle attributes are:

- Spatial localization $s_i \in \mathcal{S}$,
- mass $m_i \in \mathbb{R}$,
- charge $\mu_i \in \mathbb{R}$,
- a set of repelling particles $r_i \subseteq P$.
- The GS global state is given by the position of all the particles $\mathbf{s} = (s_1, \ldots, s_L)$.

Gravitational Swarm

Definition

The motion of the particle in the space is governed by equation:

$$\dot{s}_i(t) = -m_i(t) A_i(t) + \mu_i(t) R_i(t) + \eta(t), \tag{1}$$

where

- $A_i(t)$ attractive forces
- $R_i(t)$ repulsive forces, and
- $\eta(t)$ is a random (small) noise term.

Gravitational Swarm

Definition

The attractive motion term is of the form:

$$A_i(t) = \sum_{p_j \in P - r_i} m_j(t)(s_i - s_j)\,\delta_{ij}^A, \tag{2}$$

where

$$\delta_{ij}^A = \begin{cases} \|s_i - s_j\|^{-2} & \|s_i - s_j\|^2 > \theta_A \\ 0 & \|s_i - s_j\|^2 \leq \theta_A \end{cases}. \tag{3}$$

Gravitational Swarm

Definition

The repulsive term is of the form

$$R_i(t) = \sum_{p_j \in r_i} \mu_j(t)(s_i - s_j)\,\delta_{ij}^R.$$

where

$$\delta_{ij}^R = \begin{cases} \|s_i - s_j\|^{-2} & \|s_i - s_j\|^2 \geq \theta_R \\ 0 & \|s_i - s_j\|^2 > \theta_R \end{cases} . \quad (4)$$

Gravitational Swarm

- The attractive δ_{ij}^{A} corresponds to a strength of attraction that grows inversely to the distance between agents.
  - threshold θ_A which determines a region around the particles where the attraction forces disappear.
- The repulsive δ_{ij}^{R} defines the maximum extension of the local repulsive forces, which are short range forces.
  - The threshold θ_R determines the region around the particles where the repulsive forces are active.

Gravitational Swarm

Definition

A GS global state $\mathbf{s}(t)$ is stable when all particles have zero velocity at this state, so that $\mathbf{s}(t) = \mathbf{s}(t')$ for all $t' > t$.

- A particle p_i reaches zero velocity when it is spatially clustered with a set of particles which are non repulsive to it, and all particles repelling it are at distance greater than the specified threshold.
  - $\|s_i - s_j\|^2 \leq \theta_A$ for all $p_j \in P - r_i$, and $\|s_i - s_j\|^2 > \theta_R$ for all $p_j \in r_i$.

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Gravitational Swarm for GCP

Definition

GS-GC is a GS:

- Particles $P = P_C \cup P_V$, where
  - $P_V = \{p_1, \ldots, p_N\}$ are the vertex particles , and
  - $P_C = \{p_{N+1}, \ldots, p_{N+M}\}$ are the static color particles.
- The repulsive particles:

$$r_i = \left\{ p \in P_V \left| \phi^{-1}(p) \in \mathcal{N}\left(\phi^{-1}(p_i)\right) \right. \right\}.$$

- The mass of color particles $m_i \gg \mu_j$ for $p_i \in P_C, p_j \in P_V$.
- Color particles are static:$\dot{s}_i = 0$; $p_i \in P_C$.
- Attraction term $A_i(t) = \sum_{p_j \in P_C} \left\{ m_j (s_i - s_j) \delta_{ij}^A \right\}$.

Convergence

Theorem

A vertex particle of a GS-GC reaches zero velocity if and only if it is at distance below θ_A of a color particle and there is no repulsive particle at distance below θ_R.

Corollary

Distances between color particles must be above the repulsive range $\|s_i - s_j\|^2 > \theta_R$ for $p_i, p_j \in P_C$, $p_i \neq p_j$ to ensure that colored particles can reach zero velocity, avoiding repulsive interaction between colored particles.

Convergence

Definitions

-The neighborhood of a color particle $p_i \in P_C$ is the set of vertex particles inside its threshold of attraction
$\mathcal{N}(p_i) = \left\{ p_j \in P_V \middle| \|s_i - s_j\|^2 \leq \theta_A \middle| \right\}$.
-Color particle neighborhoods are disjoint $\mathcal{N}(p_i) \cap \mathcal{N}(p_{i'}) = \emptyset$ for any $p_i \neq p_{i'}$.
-A global state is the vector composed of all vertex particles positions $\mathbf{s} = \{s_i; p_i \in P_V\}$.
-A global state of the GS-GC is stable if all the particle velocities are simultaneously zero: $\forall p_i \in P; \dot{s}_i = 0$.

Convergence

Theorem

A global state of the GS-GC is stable if and only if all vertex particles are placed in the neighborhood of some color particle without any repulsive particles located at the same color particle neighborhood:

$$\bigcup_{p_j} \mathcal{N}\left(p_j\right) = P_V, \tag{5}$$

$$p_i \in \mathcal{N}\left(p_j\right) \Rightarrow \mathcal{N}\left(p_j\right) \cap r_i = \emptyset. \tag{6}$$

Convergence

Theorem

If the graph's chromatic number M^* is smaller than or equal to the number of color particles $M^* \leq M$, there will be a non-empty set of stable states of the GS-GC.

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Theorem

Any stable state of the GS-GC corresponds to a graph coloring.

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Convergence

Any stable state of the GS-GC corresponds to a graph coloring and that any graph coloring corresponds to a GS-GC stable state. Therefore, there are no spurious stable states.

Convergence

Theorem

If the graph's chromatic number is greater than the number of color particles, there are no stable states in the GS-GC.

Therefore the GS-GC will not converge to a stable state if the number of color particles is lower than the chromatic number. However, lack of convergence does not allow us to give any conclusion about the chromatic number of the graph, because it may be due to the dynamics of the GS-GC.

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Experimental results

Rrandomly generated planar graphs.
Tested algorithms are:

- conventional Backtracking (BT) exhaustive algorithm,
- Brelaz Degree of Saturation (DS) algorithm [?]
- the Tabu Search (TS) random local search
- the Simulated Annealing (SA) algorithm
- the Ant Colony Optimization (ACO), and
- Gravitational Swarm Intelligence (GSI) solution, with an additional mechanism to jump out of poor local minima.

Benchmark graphs

- Graph generator of benchmark graphs with specific features,
  - a known chromatic number, and
  - density (the ratio of the number of vertices to the number of edges in the graph).
- Based Kuratowski's theorem
- Five collections of 10 planar graphs, increasing the number of vertices and vertices regularly,
- Planar graphs chromatic number upper bound is 4

Benchmark graphs

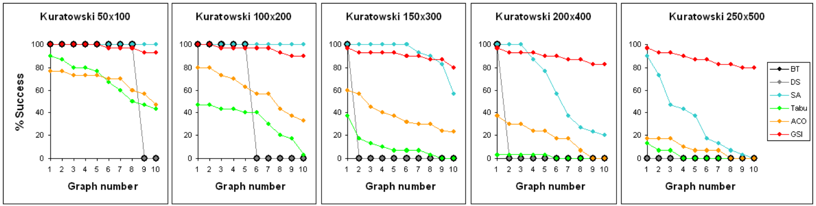
Graph name	#vertices	#Edges	Density	M^*
kuratowski 50x100	50	100	0.5	4
kuratowski 100x200	100	200	0.5	4
kuratowski 150x300	150	300	0.5	4
kuratowski 1200x400	200	400	0.5	4
kuratowski 250x500	250	500	0.5	4

Table : Features of the randomly generated planar graphs.

Experimental details

- Performance measure:
  - percentage of success measured as the number of times that the algorithm finds a correct graph coloring.
  - average time to obtain a solution measured in number of steps and in real processor time in seconds.
- Repetitions: One for backtracking and DSATUR, remaining algorithms are repeated 30 times.
- Backtracking maximum of 10^6 steps.
- Non deterministic algorithms have been allowed a maximum of 5000 steps before stopping them declaring lack of convergence, exception made of Simulated Annealing which is allowed up to 50,000 steps.

Results: success

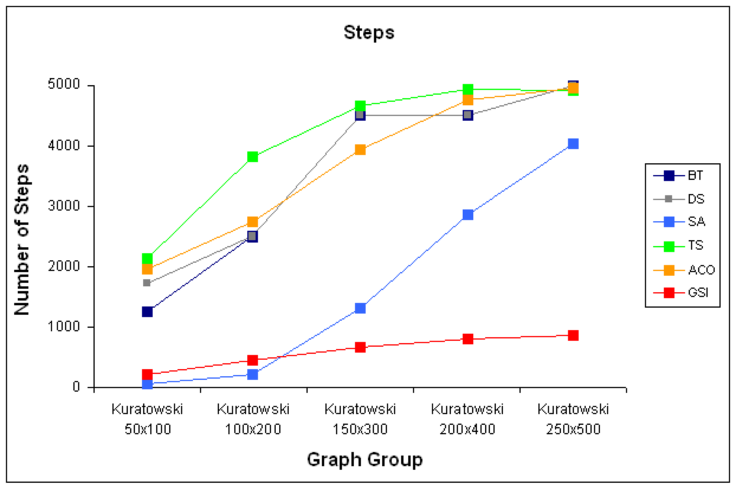


Graph coloring results for all comparing algorihtms over the graph instances. Results are ordered by success inside each collection.

Results: success

- The Backtracking and DSATUR algorithm achieved the same result for almost all the instances, consistently the worst ones.
- The GSI algorithm is among the best solutions in all cases, improving all other algorithms in many instances.
- The GSI is only improved by SA in some of the small graphs.
- The GSI has good scaling properties:
- The TS and the ACO algorithms provide suboptimal results that degrade greatly as the size of the graph increases.

Results:time steps



Results:time steps

- For SA we have normalized the number of steps to 5.000 for graphical compatibility.
- The SA seems to be the fastest for small graph instances, but the number of steps grows sharply with the graph size.
- GSI number of steps grows smoothly and linearly with the graph size. For biggest graphs it is the fastest.
- The BT, DS, TS and ACO algorithms have similar computational needs, increasing very fast with the graph size.

Results: time

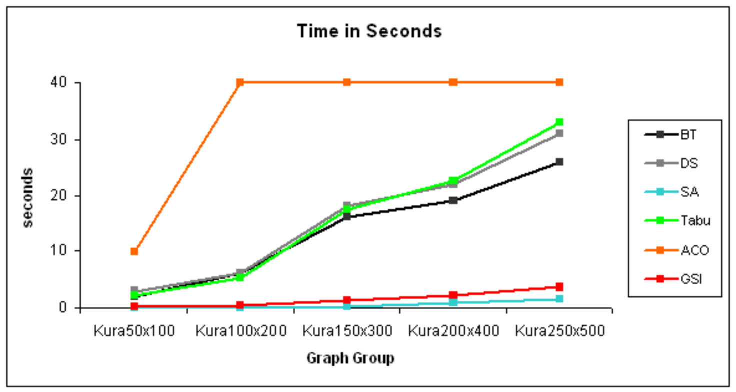


Figure : Average computational time for each collection of randomly generated graphs measured in seconds.

Results: CPU time

- In the plots we saturate at 40 seconds because the time expended by ACO grows much above the other algorithm, up to 600 seconds for the two biggest graphs.
- Both SA and GSI scale well with the problem size, growing linearly with a small slope.
- ACO is the worst case, and the other algorithms have a similar evolution of their requirements, which grow steadily with graph size.

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Conclusions

- We have introduced the Gravitational Swarm and its application to Graph Coloring Problem, the GS-GC.
- We give formal proof of some interesting results on the GS-GC stable states,
  - that they all correspond to graph colorings, and
  - that each graph coloring corresponds to a GS-GC stable state, if such states exist.
- We prove that stable states exist if the graph's chromatic number is smaller than the number of color particles defined in the GS-GC.
  - Therefore, we can use the lack of convergence as a hint about the graph's chromatic number.
- Computational experiments show competitive performance of GS-GC

Further work

- Further work is needed on the dynamic convergence of the GS-GC
    - relation betweenn the GS-GC convergence and the optimal coloring of the graph.
- Specifically, some energy-like measure associated with non-stable states is needed in order to prove global convergence of GS-GC to an stable state, and, thus, to a graph coloring.
- Top-down and bottom-up strategies for the determination of the graph's chromatic number from the system dynamics.